STATISTICAL INFERENCE FOR FOUR-REGIME SEGMENTED REGRESSION MODELS

BY HAN YAN[1,a] AND SONG XI CHEN[2,b]

[1]*Guanghua School of Management, Peking University,* [a]*hanyan@stu.pku.edu.cn*

[2]*Department of Statistics and Data Science, Tsinghua University,* [b]*sxchen@tsinghua.edu.cn*

Segmented regression models offer model flexibility and interpretability as compared to the global parametric and the nonparametric models, and yet are challenging in both estimation and inference. We consider a four-regime segmented model for temporally dependent data with segmenting boundaries depending on multivariate covariates with non-diminishing boundary effects. A mixed integer quadratic programming algorithm is formulated to facilitate the least square estimation of the regression and the boundary parameters. The rates of convergence and the asymptotic distributions of the least square estimators are obtained for the regression and the boundary coefficients, respectively. We propose a smoothed regression bootstrap to facilitate inference on the parameters and a model selection procedure to select the most suitable model within the model class with at most four segments. Numerical simulations and a case study on air pollution in Beijing are conducted to demonstrate the proposed approach, which shows that the segmented models with three or four regimes are suitable for the modeling of the meteorological effects on the $PM_{2.5}$ concentration.

1. Introduction. Regression analysis is a pivotal tool in modeling the relationship between dependent and independent variables and for prediction purposes. It is often conducted via two types of models: the global parametric and local nonparametric models. The global parametric models, such as the linear and polynomial regression models, have the advantages of interpretability and computation simplicity. However, they often perform poorly due to model misspecification as the underlying model may change over different parts of the domain. To have better adaptability, nonparametric local models facilitated by the kernel smoothing, the wavelets or splines, or the regression trees, have been introduced. The local model's complexities increase with the data's dimension and the sample sizes, elevating the risk of overfitting. The segmented model is a compromise between the global and the local models as they are as interpretable as the global parametric models but have improved model specifications.

Conventional threshold regression model (also called regime switching model) [32] was the first generation of the segmented models. It assumes that the regression function is of form $\mathbb{E}(Y|\boldsymbol{X}, Z) = \boldsymbol{X}^{\top}\boldsymbol{\beta} + \boldsymbol{X}^{\top}\boldsymbol{\delta}\mathbb{1}(Z > r)$, where Z is an observable scalar variable that can be either a time index or a pre-specified random variable. The threshold regression model has a wide range of applications in empirical research, ranging from modeling effects of shocks to economic systems over the business cycles [27], the dose-response models in biostatistics [29], and in sociological research [5]. Statistical inference of the threshold regression model with a univariate splitting variable has been well developed. [6], [15] and [16] established asymptotic properties of the least squares estimators of the threshold regression models and proposed tests on the threshold effect. As extensions, [12] and [23] introduced the multiple

MSC2020 subject classifications: Primary 62J99, 62H12; secondary 62F12.

Keywords and phrases: Mixed integer programming, Segmented model, Smoothed regression bootstrap, Temporal dependence, Threshold regression.

threshold regression model $\mathbb{E}(Y|\boldsymbol{X}, Z) = \boldsymbol{X}^{\top}\boldsymbol{\beta} + \sum_{k=1}^{K} \boldsymbol{X}^{\top}\boldsymbol{\delta}_k \mathbb{1}(Z > r_k)$ with K splits and $K+1$ regimes (segments) and investigated the statistical inference problems.

A limitation of the existing threshold regression approach is that the splitting threshold is largely determined by a univariate variable Z. [1] showed difficulties in finding the univariate splitting variable in the analysis of macroeconomic effects of fiscal policies, and [19] indicated that a univariate Z was not suitable to regulate the gene effects on disease risks as the risk of developing a particular disease was due to multiple genes. Recently, [22] and [35] extended the threshold regression to allow regime switching driven by a multivariate random vector Z which is either observable or obtained via a factor model. Although these works overcome the limitation of the univariate split variable, the setting of at most two regimes can be restrictive for some applications. Machine learning methods, such as the convex piece-wise linear fitting, can produce segmented linear regression with unlimited number of regimes. However, as these methods were focused on the fitting performances, the underlying segmented models may not be identifiable with the suggested procedures. The finite mixture models (FMM) proposed by [20] can also produce a subgroup linear model fitting for heterogeneous data. However, the subgroups from the FMMs do not lead to parameterized boundaries, and thus are less interpretable than the segmented linear models.

Our study is motivated from modelling the meteorological effects on $PM_{2.5}$ concentration in Beijing, where a global parametric model is too simple to offer good fitting performances and a nonparametric model may be too local and do not provide sufficient atmospheric interpretation. The air pollution in Beijing is typically governed by different meteorological regimes, namely the removal process by favourable northerly wind which removes $PM_{2.5}$ to a low level, the calm regime between the northerly cleaning and the start of the transported pollution driven under the southerly wind, the pollution growth regime under southerly wind that transports polluted air from the south, and air stagnation regime after the pollution has peaked, followed by the next removal process by the northerly wind. These motivate the four-regime segmented regression model in this work. As the air quality and meteorological data are time series, we consider temporally dependent data in the study.

Motivated by the air pollution problem, we consider four-regime regression models whose splitting hypersplines are determined by linear combinations of two multivariate covariates $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$, where the splitting variables $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ can be any regressors, and the two splitting hyperplanes can intersect. These make the four regime-regression model less restrictive than the multiple threshold regression model of [2] and [23] where the splitting variable Z is univariate, and hence allows not necessarily parallel boundary hyperplanes. The four-regime models include two and three regime models as special cases, where the splitting boundaries are either parallel or two adjacent regimes share the same regression coefficient and hence can be merged.

The main contributions of the study are the following. We first establish the consistency and the asymptotic distributions of the least squares estimators (LSEs) for both the boundary and the regression coefficients under the four-regime regression model with temporally dependent ρ-mixing observations, overcoming challenges posed by (i) the irregular objective function, (ii) the fixed boundary edge effects rather than the diminishing effects commonly treated in the literature and (iii) the unconventional form of the asymptotic distribution for the boundary coefficient vector. It is found that the asymptotic distribution of LSEs for the boundary coefficients is determined by the minimizers of a compound multivariate Poisson process, whose jumps depend on the points near the true hyperplanes, and the boundary coefficient estimators are asymptotically independent of the regression coefficient estimators.

The generalization to the four regimes with two splitting boundaries brings considerable computational challenges. Although the LSE of the conventional threshold regression can be obtained with the grid search method, it is not practical in our setting as the boundaries are

defined with multivariate variables. To overcome the challenges, we draw inspiration from [3] and [22] and propose an algorithm based on the mixed integer quadratic programming (MIQP), which is not only computationally efficient but also can be further accelerated by adding an iterative component. It is shown the algorithm can facilitate efficient computation of the LSEs with the rather non-regular form of the least squares objective function.

To permit statistical inference, especially in light of the rather unusual asymptotic distribution for the boundary coefficient estimates, we develop a smoothed regression bootstrap method and establish its consistency for approximating the distribution of the LSEs. Furthermore, the properties of the LSEs under degenerated segmented models with less than four regimes are investigated. In order to find the right segmented models with up to four segments, we propose a model selection method with a backward elimination procedure that is shown to be able to consistently choose the right number of regimes.

The paper is organized as follows. Section 2 introduces the four-regime regression model. Section 3 presents the theoretical properties and the asymptotic distribution of the LSEs for the regression and boundary parameters. In Section 4, we construct a mixed integer quadratic programming (MIQP) algorithm to efficiently compute the LSEs. Section 5 considers inference problems for the four-regime regression model. Section 6 investigates the properties of the proposed estimator under degenerated models with less than four regimes and proposes a model selection method. Sections 7 and 8 report simulation and empirical results, respectively. Section 9 conclude the paper with possible extensions. All technical proofs are relegated to a supplementary material (SM, [33]).

2. Model setup. We first introduce some notations. We use $\mathbb{1}(\mathcal{A})$ for the indicator function of an event $\mathcal{A}$, $\|\boldsymbol{v}\| = (\sum_{i=1}^{d} v_i^2)^{1/2}$ for the L_2-norm of vector $\boldsymbol{v} = (v_1, \cdots, v_d)^{\mathrm{T}}$ and $\mathcal{N}(\boldsymbol{v}_0; \delta) = \{\boldsymbol{v} : \|\boldsymbol{v} - \boldsymbol{v}_0\| \le \delta\}$ for the δ-neighborhood of $\boldsymbol{v}$. Define $\boldsymbol{v}_{-1}$ as the sub-vector of $\boldsymbol{v}$ excluding its first element, i.e., $\boldsymbol{v}_{-1} = (v_2, \cdots, v_d)^{\mathrm{T}}$. We use $|E|$ to denote the cardinality of a set E. For any two sets E_1 and E_2, we denote $E_1 \triangle E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$ as their symmetric difference.

Let $\{\boldsymbol{W}_t = (Y_t, \boldsymbol{X}_t, \boldsymbol{Z}_{1,t}, \boldsymbol{Z}_{2,t})\}_{t=1}^{T}$ be a sequence of observations, where Y_t is the response variable to covariates $\boldsymbol{X}_t \in \mathbb{R}^p$ and two partitioning variables $\boldsymbol{Z}_{i,t} \in \mathbb{R}^{d_i}$ for $i = 1$ and 2, which determine the boundaries of the segments or regimes. The variables $\boldsymbol{X}_t$, $\boldsymbol{Z}_{1,t}$ and $\boldsymbol{Z}_{2,t}$ can share common variables. The four-regime regression model is

$$Y_t = \sum_{k=1}^{4} \boldsymbol{X}_t^{\mathrm{T}} \boldsymbol{\beta}_{k0} \mathbb{1}\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma}_0)\} + \varepsilon_t, \tag{2.1}$$

where $\boldsymbol{Z}_t$ is the union of variables of $\boldsymbol{Z}_{1,t}$ and $\boldsymbol{Z}_{2,t}$, $\{\boldsymbol{\beta}_{k0}\}_{k=1}^{4}$ are the regression coefficients, $\{\boldsymbol{\gamma}_{i0}\}_{j=1}^{2}$ are the boundary coefficients, ε_t is the residual satisfying $\mathbb{E}(\varepsilon_t | \boldsymbol{X}_t, \boldsymbol{Z}_t) = 0$ with a finite second moment, and $R_k(\boldsymbol{\gamma}_0)$ is the k-th region split by the hyperplanes $\{H_{i0} : \boldsymbol{z}_i^{\mathrm{T}} \boldsymbol{\gamma}_{i0} = 0\}_{i=1}^{2}$ for $\boldsymbol{z}_i \in \mathbb{R}^{d_i}$. The overall parameter of interest is $\boldsymbol{\theta} = (\boldsymbol{\gamma}^{\mathrm{T}}, \boldsymbol{\beta}^{\mathrm{T}})^{\mathrm{T}}$ where $\boldsymbol{\beta} = (\boldsymbol{\beta}_1^{\mathrm{T}}, \cdots, \boldsymbol{\beta}_4^{\mathrm{T}})^{\mathrm{T}}$ and $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1^{\mathrm{T}}, \boldsymbol{\gamma}_2^{\mathrm{T}})^{\mathrm{T}}$. We let $\boldsymbol{\theta}_0$, $\boldsymbol{\beta}_0$ and $\boldsymbol{\gamma}_0$ denote the respective true parameters. For any observation $\boldsymbol{W}_t$, it is the signs of $\boldsymbol{Z}_{1,t}^{\mathrm{T}} \boldsymbol{\gamma}_{10}$ and $\boldsymbol{Z}_{2,t}^{\mathrm{T}} \boldsymbol{\gamma}_{20}$ that determine which regression region it is located at. Denote by $\mathbb{1}_1(U, V) = \mathbb{1}(U > 0, V > 0), \mathbb{1}_2(U, V) = \mathbb{1}(U \le 0, V > 0), \mathbb{1}_3(U, V) = \mathbb{1}(U \le 0, V \le 0)$ and $\mathbb{1}_4(U, V) = \mathbb{1}(U > 0, V \le 0)$. Then we can write Model (2.1) equivalently as

$$Y_t = \sum_{k=1}^{4} \boldsymbol{X}_t^{\mathrm{T}} \boldsymbol{\beta}_{k0} \mathbb{1}_k(\boldsymbol{Z}_{1,t}^{\mathrm{T}} \boldsymbol{\gamma}_{10}, \boldsymbol{Z}_{2,t}^{\mathrm{T}} \boldsymbol{\gamma}_{20}) + \varepsilon_t, \tag{2.2}$$

which explicitly reflects the role of $\boldsymbol{\gamma}_0$ in Model (2.1).

REMARK 2.1. Although the splitting hyperplanes appears linear, non-linearity may be accommodated by including nonlinear transformed variables in $\boldsymbol{Z}_i (i = 1, 2)$, for instance, $\boldsymbol{Z}_1 = (Z_1, Z_1^2, 1)^{\mathrm{T}}$. The same extension can be conducted to $\boldsymbol{X}$. It is also noted that in the special case of $\boldsymbol{Z}_{1,t}$ having the same distribution with $\boldsymbol{Z}_{2,t}$, the four segments under $\boldsymbol{\gamma}_0 = (\boldsymbol{\gamma}_{10}^{\mathrm{T}}, \boldsymbol{\gamma}_{20}^{\mathrm{T}})^{\mathrm{T}}$ are not distinguishable from that under $\tilde{\boldsymbol{\gamma}}_0 = (\boldsymbol{\gamma}_{20}^{\mathrm{T}}, \boldsymbol{\gamma}_{10}^{\mathrm{T}})^{\mathrm{T}}$. Consequently, $\boldsymbol{\theta}_0$ is only identifiable up to some permutations. To avoid such situation, we assume that the distributions of $\boldsymbol{Z}_{1,t}$ and $\boldsymbol{Z}_{2,t}$ are distinct.

REMARK 2.2. Since the signs of $\boldsymbol{Z}_1^{\mathrm{T}}\boldsymbol{\gamma}_{10}$ and $\boldsymbol{Z}_2^{\mathrm{T}}\boldsymbol{\gamma}_{20}$ determine the regimes in Model (2.1), $\boldsymbol{\gamma}_{10}$ and $\boldsymbol{\gamma}_{20}$ have to be normalized in order to be identifiable. For any candidate $\boldsymbol{\gamma}_i$ of $\boldsymbol{\gamma}_{i0}$, we normalize it by its first element $\gamma_{i,1}$, resulting in $\boldsymbol{\gamma}_i =: (1, \widetilde{\boldsymbol{\gamma}}_i)$ where $\widetilde{\boldsymbol{\gamma}}_i$ is assumed to take values in a compact set. As noted in [22], an alternative normalization is $\|\boldsymbol{\gamma}_i\|_2 = 1$. In this study, we employ the former as it has one less parameter.

3. Estimation and asymptotic properties. In this section, we outline the least squares (LS) estimation for $\boldsymbol{\theta}_0$ of the four-regime regression model, and establish the convergence rates of the LS estimators for the regression coefficient $\widehat{\boldsymbol{\beta}}$ and the boundary coefficient $\widehat{\boldsymbol{\gamma}}$ followed by providing their asymptotic distributions.

With the data sample $\{\boldsymbol{W}_t = (Y_t, \boldsymbol{X}_t, \boldsymbol{Z}_{1,t}, \boldsymbol{Z}_{2,t})\}_{t=1}^T$, in view of $\mathbb{E}(\varepsilon_t | \boldsymbol{X}_t, \boldsymbol{Z}_t) = 0$, we define the following least squares criterion function

$$\mathbb{M}_T(\boldsymbol{\theta}) = \frac{1}{T}\sum_{t=1}^{T}\left\{Y_t - \sum_{k=1}^{4}\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\beta}_k \mathbb{1}_k(\boldsymbol{Z}_{1,t}^{\mathrm{T}}\boldsymbol{\gamma}_1, \boldsymbol{Z}_{2,t}^{\mathrm{T}}\boldsymbol{\gamma}_2)\right\}^2 =: \frac{1}{T}\sum_{t=1}^{T} m(\boldsymbol{W}_t, \boldsymbol{\theta}), \tag{3.1}$$

and the parameter space is $\Theta = \Gamma_1 \times \Gamma_2 \times \mathcal{B}^4$, where Γ_i is a compact set in $\mathbb{R}^{d_i}$ and the first element of any $\boldsymbol{\gamma} \in \Gamma_i$ is normalized as 1 for each $i = 1, 2$, and $\mathcal{B}$ is a compact set in $\mathbb{R}^p$. Since $\mathbb{M}_T(\boldsymbol{\theta})$ is strictly convex in $\boldsymbol{\beta}$ and piece-wise constant in $\boldsymbol{\gamma}$ with at most T jumps, it has a unique minimizer $\widehat{\boldsymbol{\beta}} = (\widehat{\boldsymbol{\beta}}_1^{\mathrm{T}}, \cdots, \widehat{\boldsymbol{\beta}}_4^{\mathrm{T}})^{\mathrm{T}}$ for $\boldsymbol{\beta}$, but a set of minimizers for $\boldsymbol{\gamma}$, which is denoted as $\widehat{\mathcal{G}}$, such that a LSE $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\gamma}}^{\mathrm{T}}, \widehat{\boldsymbol{\beta}}^{\mathrm{T}})^{\mathrm{T}}$ satisfies

$$\mathbb{M}_T(\widehat{\boldsymbol{\theta}}) = \inf_{\boldsymbol{\theta}\in\Theta} \mathbb{M}_T(\boldsymbol{\theta}) \ \text{ for any } \ \widehat{\boldsymbol{\gamma}} \in \widehat{\mathcal{G}}. \tag{3.2}$$

It is noted that for any two $\widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\gamma}}' \in \widehat{\mathcal{G}}$, the segmented regimes under the corresponding hyperplanes must be the same, as otherwise the estimated regression coefficients will be distinct. In addition, the set $\widehat{\mathcal{G}}$ is convex since for each $i = 1$ or 2, $\boldsymbol{Z}_{i,t}^{\mathrm{T}}\widehat{\boldsymbol{\gamma}}_i > 0$ and $\boldsymbol{Z}_{i,t}^{\mathrm{T}}\widehat{\boldsymbol{\gamma}}_i' > 0$ imply that $\boldsymbol{Z}_{i,t}^{\mathrm{T}}\widetilde{\boldsymbol{\gamma}}_i > 0$ for all $\widetilde{\boldsymbol{\gamma}}_i = \alpha\widehat{\boldsymbol{\gamma}}_i + (1-\alpha)\widehat{\boldsymbol{\gamma}}_i'$ with $\alpha \in [0, 1]$. In the rest of this section, we investigate the properties of the LS estimators $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\gamma}}^{\mathrm{T}}, \widehat{\boldsymbol{\beta}}^{\mathrm{T}})^{\mathrm{T}}$ with $\widehat{\boldsymbol{\gamma}} \in \widehat{\mathcal{G}}$.

3.1. *Identification and consistency.* Here we discuss the identification of $\boldsymbol{\theta}_0$ and establish the consistency of the LSEs $\widehat{\boldsymbol{\theta}}$. Let $\boldsymbol{W} = (Y, \boldsymbol{X}, \boldsymbol{Z}_1, \boldsymbol{Z}_2)$ follow the stationary distribution $\mathbb{P}_0$ of $\boldsymbol{W}_t$, and $q_i = \boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{\gamma}_{i0}$ for $i = 1$ and 2 to indicate whether $\boldsymbol{Z} = (\boldsymbol{Z}_1, \boldsymbol{Z}_2)$ is located on the true hyperplane $H_{i0} : \boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{\gamma}_{i0} = 0$ or not. Let $\mathcal{S}(i)$ be the set consisting of index pairs (k, h) if $R_k(\boldsymbol{\gamma}_0)$ and $R_h(\boldsymbol{\gamma}_0)$ are two adjacent regions split by H_{i0}. Specifically, $\mathcal{S}(1) = \{(1,2), (2,1)(3,4), (4,3)\}$ and $\mathcal{S}(2) = \{(1,4), (4,1), (2,3), (3,2)\}$ according to the provision in the lines above (2.2). Furthermore, let $\boldsymbol{Z}$ be the union vector of variables in $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$

ASSUMPTION 1 (temporal dependence). (i) The time series $\{\boldsymbol{W}_t\}_{t\geq 1}$ is strictly stationary and ρ-mixing with the mixing coefficient $\rho(t) \leq c\alpha^t$ for finite positive constants c and

$\alpha \in (0,1)$, where $\rho(t) = \sup_{s,t\geq 1}\{\sup \mathrm{Corr}(f,g) : f \in \Omega_1^s, g \in \Omega_{s+t}^{\infty}\}$, where Ω_i^j denotes the σ-filed generated by $\{\boldsymbol{W}_t : i \leq t \leq j\}$. (ii) $\mathbb{E}(\varepsilon_t|\mathcal{F}_{t-1}) = 0$, where $\mathcal{F}_{t-1}$ is a filtration generated by $\{(\boldsymbol{X}_i, \boldsymbol{Z}_{1i}, \boldsymbol{Z}_{2i}, \varepsilon_{i-1}) : i \leq t\}$.

ASSUMPTION 2 (identification). For $i \in \{1,2\}$ and $k,h \in \{1,\cdots,4\}$, (i) $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ are not identically distributed. (ii) There exists a $j \in \{1,\cdots,d_i\}$ such that $\mathbb{P}(|q_i| \leq \epsilon|\boldsymbol{Z}_{-j,i}) > 0$ almost surely for $\boldsymbol{Z}_{-j,i}$ and for any $\epsilon > 0$, where $\boldsymbol{Z}_{-j,i}$ is the vector after excluding $\boldsymbol{Z}_i$'s jth element; without loss of generality, assume $j = 1$. (iii) For any $\boldsymbol{\gamma} \in \Gamma_1 \times \Gamma_2$ and $\mathbb{P}\{\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0) \cap R_h(\boldsymbol{\gamma})\} > 0$, the smallest eigenvalue of $\mathbb{E}\{\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}|\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0) \cap R_h(\boldsymbol{\gamma})\} \geq \lambda_0$ for some constant $\lambda_0 > 0$. (iv) For $(k,h) \in \mathcal{S}(i)$, $\|\boldsymbol{\beta}_{k0} - \boldsymbol{\beta}_{h0}\| > c_0$ for some constant $c_0 > 0$.

ASSUMPTION 3. (i) $\mathbb{E}(Y^4) < \infty$, $\mathbb{E}(\|\boldsymbol{X}\|^4) < \infty$ and $\max_{i=1,2}\mathbb{E}(\|\boldsymbol{Z}_i\|) < \infty$. (ii) For each $i = 1$ and 2, $\mathbb{P}(\boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{\gamma}_1 < 0 < \boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{\gamma}_2) \leq c_1\|\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2\|$ if $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \in \mathcal{N}(\boldsymbol{\gamma}_{i0}; \delta_0)$, for some constants $\delta_0, c_1 > 0$.

Assumption 1 (i) prescribes the strict stationarity and ρ-mixing condition on the time series, as used in the existing time-series threshold regression literature ([16] and [22]). It is noted that such a decaying rate is only required in deriving the limiting distribution of $\widehat{\boldsymbol{\gamma}}$, which can be relaxed to the polynomial decay for Theorem 3.1 and Theorem 3.2. Assumption 1 (ii) imposes a martingale difference condition for the noises, which is standard for time series regressions.

Assumption 2 is for the identification of $\boldsymbol{\theta}_0$. Specifically, without Assumption 2 (i), $(\boldsymbol{\gamma}_1^{\mathrm{T}}, \boldsymbol{\gamma}_2^{\mathrm{T}})^{\mathrm{T}}$ are not distinguishable from $(\boldsymbol{\gamma}_2^{\mathrm{T}}, \boldsymbol{\gamma}_1^{\mathrm{T}})^{\mathrm{T}}$ as discussed in Remark 2.1. It is noted that the methods and theories in the rest of the papers are applicable without such a condition, while a permutation for $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ is possibly required. Section F of the SM ([33]) provides sufficient conditions for Assumption 2 (ii), which ensures there are positive probability of observations located around the true splitting hyperplanes. Discrete variables can be accommodated in $\boldsymbol{Z}_i$, as long as it includes at least one continuous variable, say $Z_{1,i}$. Otherwise, if all the splitting variables are discretely distributed, then $\mathbb{E}\{m(\boldsymbol{W}, \boldsymbol{\theta})\}$ will be piece-wise constant and $\boldsymbol{\gamma}_0$ will not be identifiable. Assumption 2 (iii) guarantees that the splittings by candidate hyperplanes do not lead to degenerated covariance matrices, which is needed for the identification of $\boldsymbol{\beta}_0$. Assumption 2 (iv) means that adjacent regimes have distinguishable regression coefficients so that the splitting effect of each hyperplane is strictly bounded away 0, which is similar to the fixed threshold effect models treated in [6] and [35]. Assumption 3 (i) is a moment condition, and (ii) means $\mathbb{P}(\boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{\gamma} < 0)$ is continuous at $\boldsymbol{\gamma}_{i0}$, implying that $\mathbb{E}\{m(\boldsymbol{W}; \boldsymbol{\theta})\}$ is continuous at the true parameter $\boldsymbol{\theta}_0$.

The identification of $\boldsymbol{\theta}_0$ is formally ensured in the following proposition.

PROPOSITION 3.1. *Under Assumptions 1 and 2,* $\mathbb{E}\{m(\boldsymbol{W}, \boldsymbol{\theta})\} > \mathbb{E}\{m(\boldsymbol{W}, \boldsymbol{\theta}_0)\}$ *for any* $\boldsymbol{\theta} \in \Theta$ *and* $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$.

The proposition ensures that despite the multiple LS estimates $\widehat{\boldsymbol{\gamma}}$, the underlying $\boldsymbol{\gamma}_0$ is unique. The following theorem shows that any LSE estimators $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\gamma}}^{\mathrm{T}}, \widehat{\boldsymbol{\beta}}^{\mathrm{T}})^{\mathrm{T}}$ defined in (3.2) are consistent to $\boldsymbol{\theta}$. It is worth noting that though there exist infinitely many solutions $\widehat{\boldsymbol{\gamma}}$ which are collected in the convex set $\widehat{\mathcal{G}}$, the consistency of each $\widehat{\boldsymbol{\gamma}}$ can be guaranteed, which implies that the solution set $\widehat{\mathcal{G}}$ is a local neighborhood of $\boldsymbol{\gamma}_0$ with a shrinking radius.

THEOREM 3.1. *Under Assumptions 1–3, let* $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\gamma}}^{\mathrm{T}}, \widehat{\boldsymbol{\beta}}^{\mathrm{T}})^{\mathrm{T}}$ *for any* $\widehat{\boldsymbol{\gamma}} \in \widehat{\mathcal{G}}$, *, then* $\widehat{\boldsymbol{\theta}} \xrightarrow{p} \boldsymbol{\theta}_0$ *as* $T \to \infty$.

With the estimated splitting hyperplanes, each datum can be classified into one of the four estimated regimes $\{R_k(\widehat{\boldsymbol{\gamma}})\}_{k=1}^4$. Besides the estimation accuracy of $\boldsymbol{\theta}_0$, the classification accuracy is also an important criterion. It is shown next that the estimated regime $R_k(\widehat{\boldsymbol{\gamma}})$ is consistent to the true regime $R_k(\boldsymbol{\gamma}_0)$ for each $k = 1, \cdots, 4$.

COROLLARY 3.1. *Under the conditions of Theorem 3.1,* $\mathbb{P}\{\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0) \triangle R_k(\widehat{\boldsymbol{\gamma}})\} \to 0$ *as* $T \to \infty$ *for all* $k \in \{1, \cdots, 4\}$.

3.2. *Convergence rates and asymptotic distributions.* We first study the convergence rates of the LSEs $\widehat{\boldsymbol{\beta}}$ and $\widehat{\boldsymbol{\gamma}}$, which require the following conditions.

ASSUMPTION 4. (i) For $i = 1$ and 2, there exist constants $\delta_1, c_2 > 0$ such that if $\epsilon \in (0, \delta_1)$ then $\mathbb{P}(|q_i| < \epsilon | \boldsymbol{Z}_{-1,i}) \geq c_2\epsilon$ almost surely. (ii) For $i = 1$ and 2, there exists a neighborhood $\mathcal{N}_i = \mathcal{N}(\boldsymbol{\gamma}_{i0}; \delta_2)$ of $\boldsymbol{\gamma}_{i0}$ for some $\delta_2 > 0$, such that $\inf_{\boldsymbol{\gamma} \in \mathcal{N}_i} \mathbb{E}(\|\boldsymbol{X}^{\mathrm{T}}\boldsymbol{\delta}_{kh,0}\| | \boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{\gamma} = 0) > 0$ almost surely for each $(k, h) \in \mathcal{S}(i)$, where $\boldsymbol{\delta}_{kh,0} = \boldsymbol{\beta}_{k0} - \boldsymbol{\beta}_{h0}$. (iii) $\mathbb{P}(\boldsymbol{Z}_1^{\mathrm{T}}\boldsymbol{\gamma}_1 < 0 < \boldsymbol{Z}_1^{\mathrm{T}}\boldsymbol{\gamma}_2, \boldsymbol{Z}_2^{\mathrm{T}}\boldsymbol{\gamma}_3 < 0 < \boldsymbol{Z}_2^{\mathrm{T}}\boldsymbol{\gamma}_4) \leq c_3\|\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2\|\|\boldsymbol{\gamma}_3 - \boldsymbol{\gamma}_4\|$ for some constant $c_3 > 0$ if $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \mathcal{N}_1$ and $\boldsymbol{\gamma}_3, \boldsymbol{\gamma}_4 \in \mathcal{N}_2$. (iv) $\sup_{\boldsymbol{\gamma} \in \mathcal{N}_i} \mathbb{E}(\|\boldsymbol{X}\|^8 | \boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{\gamma} = 0) < \infty$ and $\sup_{\boldsymbol{\gamma} \in \mathcal{N}_i} \mathbb{E}(\varepsilon^8 | \boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{\gamma} = 0) < \infty$ almost surely.

Assumption 4 (i) strengthens Assumption 2 (i) and is satisfied when the conditional density $f_{q_i|\boldsymbol{Z}_{-1,i}}(q)$ is continuous and bounded away from 0 at $q = 0$ almost surely. Assumption 4 (ii) ensures there is a jump of the regression surface at the splitting hyperplane, which is similar to Assumption D3 of [35] and Assumption 4.(iii) of [22]. Assumption 4 (iii) controls the probability of data near the cross regions of the two hyperplanes, whose sufficient condition is presented in Section F of the SM ([33]). Assumption 4 (iv) requires that $\|\boldsymbol{X}\|$ and ε has a finite moment of the order 8 around the hyperplanes.

The next theorem establishes the rates of convergence of $\widehat{\boldsymbol{\beta}}$ and $\widehat{\boldsymbol{\gamma}}$, followed by the convergence rate of the proportions of misclassifications.

THEOREM 3.2. *Under Assumptions 1–4,* $\|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\| = O_p(1/\sqrt{T})$ *and* $\|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\| = O_p(1/T)$ *for any* $\widehat{\boldsymbol{\gamma}} \in \widehat{\mathcal{G}}$.

COROLLARY 3.2. *Under the conditions of Theorem 3.2,* $\mathbb{P}\{\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0) \triangle R_k(\widehat{\boldsymbol{\gamma}})\} = O(1/T)$ *for all* $k \in \{1, \cdots, 4\}$.

The theorem, whose proof is in Section B of the SM ([33]), shows that the regression coefficient estimator $\widehat{\boldsymbol{\beta}}$ converges to $\boldsymbol{\beta}_0$ at the standard $\sqrt{T}$-rate, while the boundary parameter estimator $\widehat{\boldsymbol{\gamma}}$, despite having multiple solutions, converges to $\boldsymbol{\gamma}_0$ at the faster T-rate. The super convergence rate attained by $\widehat{\boldsymbol{\gamma}}$ is quite typical for the boundary parameter estimators, for instance, the maximum likelihood estimator for the boundary parameter of uniform distributions, the LS estimator of models with a jump in the conditional density [8], the threshold regression model [6] and the two-regime regression model with a fixed threshold effect [35]. An intuition for the fast convergence of $\widehat{\boldsymbol{\gamma}}$ is that the discontinuity of the regression planes is highly informative for the inference of $\boldsymbol{\gamma}$. It is noted that in the shrinking threshold effect setting $\boldsymbol{\beta}_{10} - \boldsymbol{\beta}_{20} = \boldsymbol{c}T^{-\alpha}$ with $\boldsymbol{c} \neq 0$ and $0 < \alpha < \frac{1}{2}$ adopted by [16] and [22], the convergence rate of $\widehat{\boldsymbol{\gamma}}$ is slower at $T^{1-2\alpha}$.

To present the asymptotic distributions of $\widehat{\boldsymbol{\beta}}$ and $\widehat{\boldsymbol{\gamma}}$, we define for each $k \in \{1, \cdots, 4\}$,

$$B_k = \mathbb{E}\left\{\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\mathbb{1}(\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0)\right\} \text{ and } \Sigma_k = B_k^{-1}\mathbb{E}\left\{\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\varepsilon^2\mathbb{1}(\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0)\right\} B_k^{-1}.$$

Let $q_{i,t} = \boldsymbol{Z}_{i,t}^{\mathrm{T}}\boldsymbol{\gamma}_{i0}$ and $q_i = \boldsymbol{Z}_i^{\mathrm{T}}\gamma_{i0}$ for $i = 1$ and 2. Denote by $s_i^{(k)} = (-1)^{\mathbb{1}(q_i \le 0,\ \forall \boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0))}$ be the sign of q_i for $\boldsymbol{Z} = (\boldsymbol{Z}_1^{\mathrm{T}}, \boldsymbol{Z}_2^{\mathrm{T}}) \in R_k(\boldsymbol{\gamma}_0)$. For instance, $s_1^{(1)} = s_2^{(1)} = 1$ and $s_1^{(2)} = -1, s_2^{(2)} = 1$. If $R_k(\boldsymbol{\gamma}_0)$ and $R_h(\boldsymbol{\gamma}_0)$ are adjacent such that $(k,h) \in \mathcal{S}(i)$ for $i = 1$ or 2, let

$$\xi_t^{(k,h)} = \left(\boldsymbol{\delta}_{kh,0}^{\mathrm{T}}\boldsymbol{X}_t\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{kh,0} + 2\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{kh,0}\varepsilon_t\right)\mathbb{1}\left\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma}_0) \cup R_h(\boldsymbol{\gamma}_0)\right\} \tag{3.3}$$

where $\boldsymbol{\delta}_{kh,0} = \boldsymbol{\beta}_{k0} - \boldsymbol{\beta}_{h0}$. Let $\boldsymbol{Z}_{-1,i,t}$ be the random vector of $\boldsymbol{Z}_{i,t}$ excluding its first element. Suppose $(q_i, \boldsymbol{Z}_{-1,i}, \xi^{(k,h)})$ follows the stationary distribution of $(q_{i,t}, \boldsymbol{Z}_{-1,i,t}, \xi_t^{(k,h)})$. We denote $F_{q_i|\boldsymbol{Z}_{-1,i}}(q|\boldsymbol{Z}_{-1,i})$ and $F_{\xi^{(k,h)}|q_i,\boldsymbol{Z}_{-1,i}}(\xi|q_i, \boldsymbol{Z}_{-1,i})$ as the conditional distributions of q_i on $\boldsymbol{Z}_{-1,i}$ and $\xi^{(k,h)}$ on $(q_i, \boldsymbol{Z}_{-1,i})$, respectively, and the corresponding conditional densities are $f_{q_i|\boldsymbol{Z}_{-1,i}}(q|\boldsymbol{Z}_{-1,i})$ and $f_{\xi^{(k,h)}|q_i,\boldsymbol{Z}_{-1,i}}(\xi|q_i, \boldsymbol{Z}_{-1,i})$, respectively. Let $\mathcal{Z}_{-1,i}$ be the support of the distribution of $\boldsymbol{Z}_{-1,i}$. The following is needed for the weak convergence of $\widehat{\boldsymbol{\gamma}}$.

ASSUMPTION 5. (i) For $i = 1$ and 2, there exist constants $\delta_3, c_4 > 0$ such that $\mathbb{P}(|q_{i,t}| \le \delta_3, |q_{i,t+j}| \le \delta_3) \le c_4\left\{\mathbb{P}(|q_{i,t}| \le \delta_3)\right\}^2$ uniformly for $t \ge 1$ and $j \ge 1$; (ii) For each $\boldsymbol{z}_{-1,i} \in \mathcal{Z}_{-1,i}$, the conditional density $f_{q_i|\boldsymbol{Z}_{-1,i}}(q|\boldsymbol{z}_{-1,i})$ is continuous at $q = 0$ and $c_4 \le f_{q_i|\boldsymbol{Z}_{-1,i}}(0|\boldsymbol{z}_{-1,i}) \le c_5$ for some constants $c_4, c_5 > 0$; (iii) For each $\xi \in \mathbb{R}$ and $\boldsymbol{z}_{-1,i} \in \mathcal{Z}_{-1,i}$, the conditional density $f_{\xi^{(k,h)}|q_i,\boldsymbol{Z}_{-1,i}}(\xi|q_i, \boldsymbol{z}_{-1,i})$ is continuous at $q_i = 0$ and $f_{\xi^{(k,h)}|q_i,\boldsymbol{Z}_{-1,i}}(\xi|0, \boldsymbol{z}_{-1,i}) \le c_6$ for a constant $c_6 > 0$; (iv) $\mathcal{Z}_{-1,i}$ is a compact subset of $\mathbb{R}^{d_i-1}$.

Assumption 5 (i) is a non-clustering condition that states the probability of two points are both located near the splitting hyperplane H_{i0} is of a smaller order compared to that of just one point is located near H_{i0}, which curbs the clustering of extreme events and is similar to Condition C.4 of [7]. Assumption 5 (ii) and (iii) are on the conditional densities $f_{q_i|\boldsymbol{Z}_{-1,i}}$ and $f_{\xi^{(k,h)}|q_i,\boldsymbol{Z}_{-j,i}}$, respectively, which are used to characterize behaviors of the points near H_{i0}. The compactness of $\mathcal{Z}_{-1,i}$ is required by the limiting theory of point processes ([28] and [8]), which may be attained by trimming $\boldsymbol{Z}_{-1,i,t}$ or empirical quantile transformation.

The asymptotic distribution of $\widehat{\boldsymbol{\gamma}}$ needs the following stochastic process

$$D(\boldsymbol{v}) = \sum_{i=1,2}\sum_{k,h\in\mathcal{S}(i)}\sum_{\ell=1}^{\infty}\xi_{i,\ell}^{(k,h)}\mathbb{1}\left\{J_{i,\ell}^{(k,h)} + (\boldsymbol{Z}_{i,\ell}^{(k,h)})^{\mathrm{T}}\boldsymbol{v}_{-1,i} \le 0 < J_{i,\ell}^{(k,h)}\right\}, \tag{3.4}$$

for $\boldsymbol{v} = (\boldsymbol{v}_1^{\mathrm{T}}, \boldsymbol{v}_2^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{d_1+d_2}$, where $\{(\xi_{i,\ell}^{(k,h)}, \boldsymbol{Z}_{i,\ell}^{(k,h)})\}_{\ell=1}^{\infty}$ are independent copies of $(\bar{\xi}_i^{(k,h)}, \boldsymbol{Z}_{-1,i})$ with $\bar{\xi}_i^{(k,h)} \sim F_{\xi^{(k,h)}|q_i,\boldsymbol{Z}_{-1,i}}(\xi|0, \boldsymbol{Z}_{-1,i})$, and $J_{i,\ell}^{(k,h)} = \mathcal{J}_{i,\ell}^{(k,h)}/f_{q_i|\boldsymbol{Z}_{-1,i}}(0|\boldsymbol{Z}_{i,\ell}^{(k,h)})$ with $\mathcal{J}_{i,\ell}^{(k,h)} = s_i^{(k)}\sum_{n=1}^{\ell}\mathcal{E}_{i,n}^{(k,h)}$ and $\{\mathcal{E}_{i,n}^{(k,h)}\}_{n=1}^{\infty}$ are independent unit exponential variables which are independent of $\{(\xi_{i,\ell}^{(k,h)}, \boldsymbol{Z}_{i,\ell}^{(k,h)})\}_{\ell=1}^{\infty}$. Moreover, $\{(\xi_{i,\ell}^{(k,h)}, \boldsymbol{Z}_{i,\ell}^{(k,h)}, J_{i,\ell}^{(k,h)})\}_{\ell=1}^{\infty}$ are mutually independent with respect to $i = 1, 2$ and $(k,h) \in \mathcal{S}(i)$.

Let $\mathcal{G}_D = \{\boldsymbol{v}_m : D(\boldsymbol{v}_m) \le D(\boldsymbol{v}) \text{ if } \boldsymbol{v} \ne \boldsymbol{v}_m\}$ be the set of minimizers for $D(\boldsymbol{v})$. Since $D(\boldsymbol{v})$ is a piece-wise constant random function, there are infinitely many elements in $\mathcal{G}_D$. Such a phenomenon also appears in the threshold regression, where the minimizers of the process, that is a special case of (3.4), are attained in an interval, whose left endpoint is commonly used as a representative, which is not applicable to our case since $\mathcal{G}_D$ is a polyhedron. As treated in [35], we use the centroid of $\mathcal{G}_D$ as the representative. For any set $\mathcal{A}$ of

d-dimensional vectors, the centroid of $\mathcal{A}$ is $C(\mathcal{A}) = \int_{\boldsymbol{v}\in\mathcal{A}} \boldsymbol{v}d\boldsymbol{v} / \int_{\boldsymbol{v}\in\mathcal{A}} d\boldsymbol{v}$, which can be geometrically interpreted as the center of mass of the set $\mathcal{A}$. Let $\boldsymbol{\gamma}_D^c = C(\mathcal{G}_D)$ and $\widehat{\boldsymbol{\gamma}}^c = C(\widehat{\mathcal{G}})$, where $\widehat{\mathcal{G}}$ is the set for LS estimators for $\boldsymbol{\gamma}$. The former will define the limit of $\widehat{\boldsymbol{\gamma}}^c$ as shown in Theorem 3.3. Numerically, $\widehat{\boldsymbol{\gamma}}^c$ can be approximated by the average of N elements of $\widehat{\mathcal{G}}$ for a sufficiently large N. The following theorem establishes the asymptotic distributions of $\sqrt{T}(\widehat{\boldsymbol{\beta}}_k - \boldsymbol{\beta}_{k0})$ and $T(\widehat{\boldsymbol{\gamma}}^c - \boldsymbol{\gamma}_0)$.

THEOREM 3.3 (Asymptotic distribution). *Under Assumptions 1-5, we have (i)* $\sqrt{T}(\widehat{\boldsymbol{\beta}}_k - \boldsymbol{\beta}_{k0}) \xrightarrow{d} \boldsymbol{N}(0, \Sigma_k)$ *for* $k = 1, \cdots, 4$ *and* $T(\widehat{\boldsymbol{\gamma}}^c - \boldsymbol{\gamma}_0) \xrightarrow{d} \boldsymbol{\gamma}_D^c$*; (ii)* $\{\sqrt{T}(\widehat{\boldsymbol{\beta}}_k - \boldsymbol{\beta}_{k0})\}_{k=1}^4$ *and* $\{T(\widehat{\boldsymbol{\gamma}}_i^c - \boldsymbol{\gamma}_{i0})\}_{i=1}^2$ *are asymptotically independent.*

REMARK 3.1. The limiting process $D(\boldsymbol{v})$ is derived by the asymptotics of the point process induced by $\{(\xi_t^{(k,h)}, \boldsymbol{Z}_{-1,i,t}, Tq_{i,t})\}_{t=1}^T$. The process $D(\boldsymbol{v})$ can be regarded as a multivariate compound Poisson process, whose jump sizes are $\{\xi_{i,\ell}^{(k,h)}\}_{\ell=1}^\infty$ and jump locations are determined by the counting measure induced by $\{(J_{i,\ell}^{(k,h)}, \boldsymbol{Z}_{i,\ell}^{(k,h)})\}_{\ell=1}^\infty$. Intuitively, this is because $D(\boldsymbol{v})$ largely relies on those points lying in a local neighborhood of the true splitting hyperplanes, whose $|q_{i,t}|$ are on the order of $O(T^{-1})$, which are rare events with their occurrences asymptotically governed by a Poisson process. In the case of univariate threshold model where $\boldsymbol{Z}_i = (Z, 1)^{\mathrm{T}}$ and $\boldsymbol{\gamma}_{i0} = (1, \gamma_{i0})^{\mathrm{T}}$ so that $\boldsymbol{Z}_{-1,i} = 1$ and $q_i = Z - \gamma_{i0}$, it can be seen that $D(\boldsymbol{v})$ coincides with the compound Poisson process established in [6]. Theorem 3.3 also extends the result of [35] to accommodate the temporal-dependent data and multiple splitting hyperplanes. The analysis is technically more involved than the existing literature of the fixed effect threshold regression due to the challenge of the multivariate boundaries and the dependence of the observations. To tackle these challenges, we exploit large sample theory for the extreme values and point processes ([25] and [28]), as well as the epi-convergence in distribution ([21]), which is more general than the classic uniform convergence in distribution and allows for more general discontinuity, as outlined in the SM ([33]). The techniques used in the proof may be used to analyze the asymptotic of other extreme type statistics that can be expressed as some functional of a multivariate point process with temporal-dependent sequences.

REMARK 3.2. The asymptotic independence of $T(\widehat{\boldsymbol{\gamma}}_1^c - \boldsymbol{\gamma}_{10})$ and $T(\widehat{\boldsymbol{\gamma}}_2^c - \boldsymbol{\gamma}_{20})$ was shown for the univaraite multiple-regime threshold model ([23]). Theorem 3.3 reveals that this can be extended to multiple splitting hyperplanes, provided that the probability of data locating at the crossing region of the two hyperplanes is negligible as reflected in Assumption 4 (iii). As shown in the proof, the empirical point process induced by $\{(\xi_t^{(k,h)}, \boldsymbol{Z}_{-1,i,t}, Tq_{i,t}), i = 1, 2, (k,h) \in \mathcal{S}(i)\}_{t=1}^T$ is asymptotic Poisson, whose arrivals can be divided into different segments, depending on whether they belong to the same pair $(k,h) \in \mathcal{S}(i)$ or not, where $\mathcal{S}(i)$ is the set of index pairs of adjacent regions split by the i-th hyperplane. Hence, the limiting Poisson process can be thinned into several asymptotic independent child processes, which further implies the asymptotic independence of $T(\widehat{\boldsymbol{\gamma}}_1^c - \boldsymbol{\gamma}_{10})$ and $T(\widehat{\boldsymbol{\gamma}}_2^c - \boldsymbol{\gamma}_{20})$. As a building block, we established a thinning theorem for Poisson processes for the α-mixing sequences, which might be useful in its own right. The asymptotic independence of $\sqrt{T}(\widehat{\boldsymbol{\beta}}_k - \boldsymbol{\beta}_{k0})$ and $T(\widehat{\boldsymbol{\gamma}}^c - \boldsymbol{\gamma}_0)$ can be explained by the fact that the former is asymptotically a sum of terms with each term being asymptotically negligible. Hence $\sqrt{T}(\widehat{\boldsymbol{\beta}}_k - \boldsymbol{\beta}_{k0})$ should not depend on the stochastically bounded number of points near the hyperplanes that determine the distribution of $T(\widehat{\boldsymbol{\gamma}}^c - \boldsymbol{\gamma}_0)$ ([18]).

It is also noted that the temporal dependence structure of the observed time series does not show up in the asymptotic distributions of $T(\widehat{\boldsymbol{\gamma}}^c - \boldsymbol{\gamma}_0)$ and $\sqrt{T}(\widehat{\boldsymbol{\beta}}_k - \boldsymbol{\beta}_{k0})$. That regarding $\sqrt{T}(\widehat{\boldsymbol{\beta}}_k - \boldsymbol{\beta}_{k0})$ is due to the martingale difference condition $\mathbb{E}(\varepsilon_t|\mathcal{F}_{t-1}) = 0$ as far as the asymptotic variance of $\widehat{\boldsymbol{\beta}}_k$ is concerned, which is commonly the case in other related studies [6, 23]. That on the $T(\widehat{\boldsymbol{\gamma}}^c - \boldsymbol{\gamma}_0)$ is because the asymptotic distribution of $\widehat{\boldsymbol{\gamma}}^c$ is determined by the empirical point process induced by the points near the underlying splitting hyperplanes, which satisfies Meyer's condition ([25]) for rare events of mixing sequences and ensures the limiting process being Poisson as in the case of independent observations.

4. Computation. The computation of the LSE for $\widehat{\boldsymbol{\theta}}$ by minimizing (3.2) is quite challenging due to the non-regularity of $m(\boldsymbol{W}_t, \boldsymbol{\theta})$ that makes the most commonly used optimization algorithms unworkable. We overcome the difficulty via the mixed integer quadratic programming (MIQP), which optimizes a quadratic objective function with linear constraints over points in polyhedral sets whose components can be both integer and continuous variables; see [4] and [3] for details. For the two-regime regression, [22] expressed the LS problem as an MIQP problem to improve the computation efficiency. The inclusion of the second boundary in the current study brings challenges. If formulated directly using the approach of [22], it would make the objective function quartic rather than quadratic. We will formulate a MIQP for the two-boundary problem to facilitate the computation.

To make the notations compact, we define $I_{k,t} = \mathbb{1}\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma})\}$ for any candidate $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1^{\mathrm{T}}, \boldsymbol{\gamma}_2^{\mathrm{T}})^{\mathrm{T}}$ and $k = 1, \cdots, 4$. Let $X_{t,i}$ be the i-th element of $\boldsymbol{X}_t$ and $\beta_{k,i}$ be the i-th element of $\boldsymbol{\beta}_k$. It can be noted that the irregularity of $\mathbb{M}_T(\boldsymbol{\theta})$ in (3.2) is brought by the indicators $\{I_{k,t}\}$. If we define $\ell_{k,i,t} = I_{k,t}\beta_{k,i}$ for $i = 1, \cdots, p$, then $\mathbb{M}_T(\boldsymbol{\theta})$ can be expressed as

$$\mathbb{V}_T(\boldsymbol{\ell}) = \frac{1}{T}\sum_{t=1}^{T}\left(Y_t - \sum_{k=1}^{4}\sum_{i=1}^{p} X_{t,i}\ell_{k,i,t}\right)^2 \tag{4.1}$$

which is quadratic with respect to $\boldsymbol{\ell} = \{\ell_{k,i,t} : k = 1, \cdots, 4;\ i = 1, \cdots, p;\ t = 1, \cdots, T\}$.

Since the constraints of an MIQP have to be linear, while $\ell_{k,i,t} = I_{k,t}\beta_{k,i}$ is non-linear, it is necessary to introduce linear constraints to ensure that $\{\ell_{k,i,t}\}$ have a one-to-one correspondence to the unknown parameters $\{\boldsymbol{\beta}_k\}_{k=1}^4$ and$\{\boldsymbol{\gamma}_j\}_{j=1}^2$. As $\boldsymbol{\beta}_k$ belongs to a compact set, there exist constants L_i and U_i such that $L_i \le \beta_{k,i} \le U_i$. By imposing constraints

$$I_{k,t}L_i \le \ell_{k,i,t} \le I_{k,t}U_i \quad \text{and} \quad L_i(1 - I_{k,t}) \le \beta_{k,i} - \ell_{k,i,t} \le U_i(1 - I_{k,t}), \tag{4.2}$$

it can be verified that (4.2) holds if and only if $\ell_{k,i,t} = I_{k,t}\beta_{k,i}$ under the condition that $I_{k,t} \in \{0, 1\}$. That $\ell_{k,i,t} = I_{k,t}\beta_{k,i}$ implies (4.2) is obvious. To appreciate the other way, note that if $I_{k,t} = 1$, $\ell_{k,i,t} = \beta_{k,i}$; otherwise if $I_{k,t} = 0$, $\ell_{k,i,t} = 0$. In either cases, $\ell_{k,i,t} = I_{k,t}\beta_{k,i}$.

The next goal is to relate $I_{k,t} = \mathbb{1}\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma})\}$ to the boundary coefficients $\{\boldsymbol{\gamma}_j\}_{j=1}^2$. Let $g_{j,t} = \mathbb{1}(\boldsymbol{Z}_{j,t}^{\mathrm{T}}\boldsymbol{\gamma}_j > 0)$. We first express $g_{j,t}$ by linear constraints in $\boldsymbol{\gamma}_j$, so as to link $I_{k,t}$ with $g_{j,t}$ via linear inequalities. Let $M_{j,t} = \max_{\boldsymbol{\gamma}\in\Gamma_j} |\boldsymbol{Z}_{j,t}^{\mathrm{T}}\boldsymbol{\gamma}|$ which can be readily computed via linear programming. Then,

$$(g_{j,t} - 1)(M_{j,t} + \epsilon) < \boldsymbol{Z}_{j,t}^{\mathrm{T}}\boldsymbol{\gamma}_j \le g_{j,t}M_{j,t} \tag{4.3}$$

hold by the definition of $g_{j,t}$, where $\epsilon > 0$ is a small predetermined constant. On the other hand, let $g_{j,t}$ be a binary variable that satisfies (4.3). Then, $g_{j,t} = 1$ and the first inequality implies that $Z_{j,t}^{\mathrm{T}}\boldsymbol{\gamma}_k > 0$; and $g_{j,t} = 0$ and the second inequality implies that $Z_{j,t}^{\mathrm{T}}\boldsymbol{\gamma} \le 0$. Thus, (4.3) are equivalent to $g_{j,t} = \mathbb{1}(\boldsymbol{Z}_{j,t}^{\mathrm{T}}\boldsymbol{\gamma}_j > 0)$.

Finally, we construct constraints which are linear in $\{g_{j,t}\}_{j=1}^2$ and equivalent to $I_{k,t} = \mathbb{1}\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma})\}$. Since each regime $R_k(\boldsymbol{\gamma})$ can be written as $R_k(\boldsymbol{\gamma}) = \{(\boldsymbol{z}_1, \boldsymbol{z}_2) : s_j^{(k)}\boldsymbol{z}_j^{\mathrm{T}}\boldsymbol{\gamma}_j >$

$0,\ j=1,2\}$, where $s_j^{(k)} \in \{-1,1\}$ is the sign of $\boldsymbol{z}_j^{\mathrm{T}}\boldsymbol{\gamma}_j$ for the points belonging in $R_k(\boldsymbol{\gamma})$, we can write $I_{k,t} = \prod_{j=1}^{2} \mathbb{1}(s_j^{(k)} \boldsymbol{Z}_{j,t}^{\mathrm{T}}\boldsymbol{\gamma}_j > 0)$, which can be linked to $\{g_{j,t}\}_{j=1}^{2}$ via

$$I_{k,t} = \prod_{j=1}^{2} \mathbb{1}\left(s_j^{(k)} \boldsymbol{Z}_{j,t}^{\mathrm{T}}\boldsymbol{\gamma}_j > 0\right) = \prod_{j=1}^{2} \left\{s_j^{(k)} g_{j,t} + (1 - s_j^{(k)})/2\right\}, \tag{4.4}$$

where the first equality is by the definition of $I_{k,t}$, and the second equality can be directly verified. Since the right-hand side of (4.4) is a product of two factors taking values in $\{0,1\}$, it can be shown that (4.4) is equivalent to the following linear constraints

$$I_{k,t} \geq \sum_{j=1}^{2} \left\{s_j^{(k)} g_{j,t} + (1 - s_j^{(k)})/2\right\} - 1 \ \text{ and } \ I_{k,t} \leq s_j^{(k)} g_{j,t} + (1 - s_j^{(k)})/2. \tag{4.5}$$

for $j = 1$ and 2 and $k \in \{1, \cdots, 4\}$.

In summary, via the linear constraints (4.2), (4.3) and (4.5), we transform the original LS problem (2.2) to a MIQP problem formulated as following.

Let $\boldsymbol{g} = \{g_{j,t} : j = 1,2, t = 1, \cdots, T\}$, $\boldsymbol{\mathcal{I}} = \{I_{k,t} : k = 1, \cdots, 4, t = 1, \cdots, T\}$ and $\boldsymbol{\ell} = \{\ell_{k,i,t} : k = 1, \cdots, 4, i = 1, \cdots, p, t = 1, \cdots, T\}$. Solve the following problem:

$$\min_{\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{g},\boldsymbol{\mathcal{I}},\boldsymbol{\ell}} \frac{1}{T}\sum_{t=1}^{T}\left(Y_t - \sum_{k=1}^{4}\sum_{i=1}^{p} X_{t,i}\ell_{k,i,t}\right)^2 \tag{4.6}$$

$$\text{subject to } \begin{cases} \boldsymbol{\beta}_k \in \mathcal{B},\ \ \boldsymbol{\gamma}_j \in \Gamma_j,\ \ g_{j,t} \in \{0,1\},\ \ I_{k,t} \in \{0,1\},\ \ L_i \leq \beta_{k,i} \leq U_i, \\ (g_{j,t} - 1)(M_{j,t} + \epsilon) < \boldsymbol{Z}_{j,t}^{\mathrm{T}}\boldsymbol{\gamma}_j \leq g_{j,t} M_{j,t},\ I_{k,t}L_i \leq \ell_{k,i,t} \leq I_{k,t}U_i, \\ L_i(1 - I_{k,t}) \leq \beta_{k,i} - \ell_{k,i,t} \leq U_i(1 - I_{k,t}), \\ I_{k,t} \leq s_j^{(k)} g_{j,t} + (1 - s_j^{(k)})/2,\ \ I_{k,t} \geq \sum_{j=1}^{2}\left\{s_j^{(k)} g_{j,t} + (1 - s_j^{(k)})/2\right\} - 1, \end{cases} \tag{4.7}$$

for $k = 1, \cdots, 4, j = 1, 2, i = 1, \cdots, p$ and $t = 1, \cdots, T$.

The above optimization problem can be solved quite efficiently with modern mixed integer optimization softwares such as GUROBI and CPLEX. The next theorem, whose proof is in Section C of the SM ([33]), shows that the formulated MIQP is equivalent to the original LS problem.

THEOREM 4.1. *For any small $\epsilon > 0$ in (4.7), let $\tilde{\boldsymbol{\theta}} = (\tilde{\boldsymbol{\gamma}}^{\mathrm{T}}, \tilde{\boldsymbol{\beta}}^{\mathrm{T}})^{\mathrm{T}}$ be a solution of the MIQP defined with (4.6) and (4.7), then $\mathbb{M}_T(\widehat{\boldsymbol{\theta}}) = \mathbb{M}_T(\tilde{\boldsymbol{\theta}})$ where $\widehat{\boldsymbol{\theta}}$ is a solution in (3.2).*

Theorem 4.1 indicates that any $\widetilde{\boldsymbol{\gamma}}$ satisfying (4.6) and (4.7) is an element of $\widehat{\mathcal{G}}$, the solution set for the LS estimators for $\boldsymbol{\gamma}_0$. Since for any $\{g_{j,t}\} \in \{0,1\}^{2T}$, there are infinitely many $\boldsymbol{\gamma}_j$ $(j = 1,2)$ that satisfy the constraint in the second line of (4.7), we can output multiple solutions $\{\widetilde{\boldsymbol{\gamma}}_n = (\widetilde{\boldsymbol{\gamma}}_{n1}^{\mathrm{T}}, \widetilde{\boldsymbol{\gamma}}_{n2}^{\mathrm{T}})^{\mathrm{T}}\}_{n=1}^{N}$ of the above MIQP for a sufficiently large N, and use their average as an approximation for the centroid $\widehat{\boldsymbol{\gamma}}^c$ of the set $\widehat{\mathcal{G}}$ as advocated in [35]. We display a scatter plot of the multiple solutions from a simulation experiment reported in Section H.2 of the SM ([33]), which appeared to be uniformly distributed. However, it requires further investigation to understand the detailed mechanism regarding how the multiple elements of $\widehat{\mathcal{G}}$ are produced by the MIQP solver.

REMARK 4.1. It is noted that the above algorithm requires prior specifications of (L_i, U_i), the upper and lower bound for $\beta_{k,i}$. In practice, we can first standardize $\{\boldsymbol{X}_t\}_{t=1}^T$ and specify a sufficient large parameter interval (L_i, U_i) to ensure it contains the true value. Alternatively, we can employ the data-driven method proposed in [3] that estimates $\max\{|L_i|, |U_i|\}$ via the convex quadratic optimization. Besides the proposed MIQP algorithm, the MCMC-based method as used in [35] for the two-regime regression can also be adapted to minimize the LS criterion $\mathbb{M}_T(\boldsymbol{\theta})$, which avoids the specification of the parameter bounds but requires more intensive computations since it is a simulation-based method. A comprehensive comparison between the MIQP and MCMC algorithms for segmented regressions would require more work and we leave it to further study.

REMARK 4.2. As indicated in [22], the MIQP may be slow when the dimension of $\boldsymbol{X}_t$ and the sample size T are large. As an alternative, we present a block coordinate descent (BCD) algorithm for the four-regime model in Section C of the SM ([33]), which minimizes the LS criterion with respect to $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ iteratively. At each step, the update for $\boldsymbol{\gamma}$ given $\boldsymbol{\beta}$ is via a mixed integer linear programming (MILP), which is easier to solve than the MIQP. The update for $\boldsymbol{\beta}$ given $\boldsymbol{\gamma}$ is by linear regression in each candidate regime. Hence, the BCD is computationally more efficient than the MIQP that jointly optimizes $(\boldsymbol{\gamma}, \boldsymbol{\beta})$. However, there is no guarantee that the BCD converges to the global optimal solution without a consistent initialization. Simulations to compare the two algorithms are presented in the SM ([33]), which show that the BCD with proper initial values can produce close solutions to that of the MIQP with significantly reduced running time.

5. Smoothed regression bootstrap. We now consider the statistical inference problems for $\boldsymbol{\beta}_0$ and $\boldsymbol{\gamma}_0$. The inference for $\boldsymbol{\beta}_0$ is quite standard due to the asymptotic normality of $\widehat{\boldsymbol{\beta}}$, while that for the boundary coefficient $\boldsymbol{\gamma}_0$ is much more challenging since the asymptotic distribution of $T(\widehat{\boldsymbol{\gamma}}^c - \boldsymbol{\gamma}_0)$ has a much-involved form and is hard to simulate.

A natural idea for the inference of $\boldsymbol{\gamma}_0$ is to employ the bootstrap. However, as shown in [31] and [34], neither the nonparametric, the residual, nor the wild bootstrap is consistent in approximating the distribution of estimator for the change points in change point models or the threshold in threshold regression models. The failure of these bootstrap methods can be explained as follows. As pointed out in Remark 3.2, only the data around the boundary hyperplanes is informative for the inference on $\boldsymbol{\gamma}_0$. Thus the bootstrap sampling distribution $\widehat{\mathbb{P}}_T$, when conditional on the original data, must approximate the true distribution $\mathbb{P}_0$ in the neighborhood of the true hyperplanes. For the identification of $\boldsymbol{\gamma}_0$, $\mathbb{P}_0$ must have a positive probability on any local region around the underlying boundaries, as reflected in Assumption 2 (ii). However, conditional on the original data, the bootstrap distribution $\widehat{\mathbb{P}}_T$ is discrete under either the nonparametric, the residual, or the wild bootstrap, which fails to mirror $\mathbb{P}_0$. As a remedy, we present a smoothed regression bootstrap method and prove its theoretical validity.

Suppose that Y is generated according to the following segmented linear regression model with heteroscedastic error

$$Y = \sum_{k=1}^{4} \boldsymbol{X}^{\mathrm{T}} \boldsymbol{\beta}_0 \mathbb{1}\{\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0)\} + \sigma_0(\boldsymbol{X}, \boldsymbol{Z})\, e, \tag{5.1}$$

where e has a continuous distribution and is independent of $(\boldsymbol{X}, \boldsymbol{Z})$ with $\mathbb{E}(e) = 0$ and $\mathbb{E}(e^2) = 1$, and $\sigma_0^2(\boldsymbol{X}, \boldsymbol{Z})$ is a conditional variance function representing possible heteroskedasticity. Model (5.1) is a refinement of Model (2.1) with more detailed structure on the residuals. If it is believed that the error is homogeneous within each region $R_k(\boldsymbol{\gamma}_0)$ so

that $\varepsilon = \sigma_k \mathbb{1}\{\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0)\}e$ for some $\sigma_k > 0$, as assumed in [34], then the nonparametric estimation for $\sigma_0(\boldsymbol{x}, \boldsymbol{z})$ is not required and σ_k can be estimated with the sample standard deviation of the fitted residuals in the k-th region.

Let $F_0(\boldsymbol{x}, \boldsymbol{z})$ be the distribution function of $(\boldsymbol{X}, \boldsymbol{Z})$, whose density function is $f_0(\boldsymbol{x}, \boldsymbol{z})$. We estimate $F_0(\boldsymbol{x}, \boldsymbol{z})$ and $\sigma_0(\boldsymbol{x}, \boldsymbol{z})$ nonparametrically with the kernel smoothing. Specifically, let $K_1(\cdot)$ and $K_2(\cdot)$ be a p-dimensional and a $(d_1 + d_2)$-dimensional kernel functions, respectively. Let $G_i(\boldsymbol{u}) = \int_{-\infty}^{\boldsymbol{u}} K_i(\boldsymbol{u})d\boldsymbol{u}$ for $i = 1, 2$. The kernel smoothing estimator for $F_0(\boldsymbol{x}, \boldsymbol{z})$ is given by

$$\widetilde{F}_0(\boldsymbol{x}, \boldsymbol{z}) = \frac{1}{T}\sum_{t=1}^{T} G_1\left(\frac{\boldsymbol{X}_t - \boldsymbol{x}}{h_1}\right) G_2\left(\frac{\boldsymbol{Z}_t - \boldsymbol{z}}{h_2}\right),$$

where h_1 and h_2 are smoothing bandwidths.

With the LS estimator $(\widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\beta}})$, the estimated residuals are $\widehat{\varepsilon}_t = Y_t - \sum_{k=1}^{4} \boldsymbol{X}^{\mathrm{T}}\widehat{\boldsymbol{\beta}}\mathbb{1}\{\boldsymbol{Z}_t \in R_k(\widehat{\boldsymbol{\gamma}})\}$. The conditional variance function $\sigma_0^2(\boldsymbol{x}, \boldsymbol{z})$ can be estimated via the local linear approach proposed by [10]. For any given $(\boldsymbol{x}, \boldsymbol{z})$, the local linear estimator $\widetilde{\sigma}^2(\boldsymbol{x}, \boldsymbol{z}) = \widehat{\alpha}$, which is defined by

$$(\widehat{\alpha}, \widehat{\boldsymbol{\eta}}) = \underset{(\alpha, \boldsymbol{\eta})}{\arg\min} \sum_{t=1}^{T} \left\{\widehat{\varepsilon}_t^2 - \alpha - ((\boldsymbol{X}_t - \boldsymbol{x})^{\mathrm{T}}, (\boldsymbol{Z}_t - \boldsymbol{z})^{\mathrm{T}})\,\boldsymbol{\eta}\right\}^2 K_1\left(\frac{\boldsymbol{X}_t - \boldsymbol{x}}{b_1}\right) K_2\left(\frac{\boldsymbol{Z}_t - \boldsymbol{z}}{b_2}\right),$$

where $\boldsymbol{\eta} \in \mathbb{R}^{p+d_1+d_2}$, and b_1 and b_2 are smoothing bandwidths. Let $\widehat{e}_t = \widehat{\varepsilon}_t/\widetilde{\sigma}(\boldsymbol{X}_t, \boldsymbol{Z}_t)$ and $\tilde{e}_t = \widehat{e}_t - \bar{e}_T$, where $\bar{e}_T = \sum_{t=1}^{T} \widehat{e}_t/T$. Denote $\widehat{G}(e)$ as the empirical distribution of $\{\tilde{e}_t\}_{t=1}^{T}$.

We need the following conditions on the underlying stationary distribution and its density functions, the kernel functions, and the smoothing bandwidths to facilitate the Bootstrap procedure.

ASSUMPTION 6. (i) The stationary distribution F_0 of $(\boldsymbol{X}_t, \boldsymbol{Z}_t)$ has a compact support and is absolute continuous with density $f_0(\boldsymbol{x}, \boldsymbol{z})$ which is bounded and $\inf_{\boldsymbol{x},\boldsymbol{z}} f_0(\boldsymbol{x}, \boldsymbol{z}) > 0$.

(ii) The conditional variance function $\sigma_0^2(\boldsymbol{x}, \boldsymbol{z})$ is bounded and $\inf_{\boldsymbol{x},\boldsymbol{z}} \sigma_0^2(\boldsymbol{x}, \boldsymbol{z}) > 0$.

(iii) The kernels $K_1(\cdot)$ and $K_2(\cdot)$ are symmetric density functions which are Lipshitz continuous and have bounded supports. The smoothing bandwidths satisfy $h_i, b_i \to 0$ for $i = 1$ and 2, and $T(\log T)^{-1}h_1^p h_2^{d_1+d_2} \to \infty$ and $T(\log T)^{-1}b_1^p b_2^{d_1+d_2} \to \infty$ as $T \to \infty$.

Under Assumptions 1 and 6, it can be shown that $\sup_{\boldsymbol{x},\boldsymbol{z}} \|\widetilde{F}_0(\boldsymbol{x}, \boldsymbol{z}) - F_0(\boldsymbol{x}, \boldsymbol{z})\| \xrightarrow{p} 0$, and $\sup_{\boldsymbol{x},\boldsymbol{z}} \|\widetilde{\sigma}^2(\boldsymbol{x}, \boldsymbol{z}) - \sigma_0^2(\boldsymbol{x}, \boldsymbol{z})\| \xrightarrow{p} 0$, following the uniform convergence results of kernel density and regression estimators for mixing sequences, say [14]. In addition, the above assumptions also ensure the uniform convergence of the density $\widetilde{f}_0$ of the kernel estimator $\widetilde{F}_0$ to the true density function f_0, which is required in establishing the consistency of the smoothed regression bootstrap. If $(\boldsymbol{X}, \boldsymbol{Z})$ is of high dimensions we can also employ machine learning methods that are adaptive to high dimensional features, such as the deep neural networks, to estimate $f_0(\boldsymbol{x}, \boldsymbol{z})$ and $\sigma_0(\boldsymbol{x}, \boldsymbol{z})$, as long as their uniform convergence can be guaranteed.

The bootstrap procedure to approximate the distributions of $\{T(\widehat{\boldsymbol{\gamma}}^c - \boldsymbol{\gamma}_0), \sqrt{T}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)\}$ is as follows.

Step 1: First, generate $\{(\boldsymbol{X}_t^*, \boldsymbol{Z}_t^*)\}_{t=1}^{T}$ independently from $\widetilde{F}(\boldsymbol{x}, \boldsymbol{z})$ and $\{e_t^*\}_{t=1}^{T}$ independently from $\widehat{G}(e)$, respectively. Then, generate $Y_t^* = \sum_{k=1}^{4} (\boldsymbol{X}_t^*)^{\mathrm{T}}\widehat{\boldsymbol{\beta}}_k \mathbb{1}\{\boldsymbol{Z}_t^* \in R_k(\widehat{\boldsymbol{\gamma}}^c)\} + \widetilde{\sigma}(\boldsymbol{X}_t^*, \boldsymbol{Z}_t^*)e_t^*$ to obtain bootstrap resample $\{(Y_t^*, \boldsymbol{X}_t^*, \boldsymbol{Z}_t^*)\}_{t=1}^{T}$.

Step 2: Compute the LSEs based on $\{(Y_t^*, \boldsymbol{X}_t^*, \boldsymbol{Z}_t^*)\}_{t=1}^T$, where $\widehat{\boldsymbol{\beta}}^*$ is the LSE for $\boldsymbol{\beta}_0$ and $\{\widehat{\boldsymbol{\gamma}}_i^*\}_{i=1}^N$ are the LSEs for $\boldsymbol{\gamma}_0$ for a sufficiently large N. Let $\widehat{\boldsymbol{\gamma}}^{*c} = \sum_{i=1}^N \widehat{\boldsymbol{\gamma}}_i^*/N$.

Step 3: Repeat the above two steps B times for a large positive integer B to obtain $\{\widehat{\boldsymbol{\gamma}}_b^{*c}\}_{b=1}^B$ and $\{\widehat{\boldsymbol{\beta}}_b^*\}_{b=1}^B$, and use the empirical distribution of $\left\{T(\widehat{\boldsymbol{\gamma}}_b^{*c} - \widehat{\boldsymbol{\gamma}}^c), \sqrt{T}(\widehat{\boldsymbol{\beta}}_b^* - \widehat{\boldsymbol{\beta}})\right\}_{b=1}^B$ as an estimate of the distribution of $\{T\left(\widehat{\boldsymbol{\gamma}}^c - \boldsymbol{\gamma}_0\right), \sqrt{T}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)\}$.

As in the original LS problem, the LSEs for $\boldsymbol{\gamma}_0$ based on each bootstrap resample are attained on a convex set $\widehat{\mathcal{G}}^*$. Therefore, in Step 2 we approximate the centroid of $\widehat{\mathcal{G}}^*$ by the average of N elements in $\widehat{\mathcal{G}}^*$. Denote the distribution of $\{T\left(\widehat{\boldsymbol{\gamma}}^c - \boldsymbol{\gamma}_0\right), \sqrt{T}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)\}$ as $\mathcal{L}_T$ and the empirical distribution of $\left\{T(\widehat{\boldsymbol{\gamma}}_b^{*c} - \widehat{\boldsymbol{\gamma}}^c), \sqrt{T}(\widehat{\boldsymbol{\beta}}_b^* - \widehat{\boldsymbol{\beta}})\right\}_{b=1}^B$ as $\mathcal{L}_{T,B}$. The validity of the smoothed regression bootstrap is established in the following theorem.

THEOREM 5.1. *Suppose that Assumptions 1-6 hold. Then* $\rho\left(\mathcal{L}_{T,B}, \mathcal{L}_T\right) \xrightarrow{p} 0$ *as* $B, T \to \infty$, *for any metric* ρ *that metrizes weak convergence of distributions.*

The proof of the theorem is in Section D of the SM ([33]) by first establishing sufficient conditions for a consistent bootstrap scheme for approximating $\mathcal{L}_T$, followed by showing that the smoothed regression bootstrap satisfies these conditions. With the above result, confidence regions and hypothesis testings about $\boldsymbol{\gamma}_0$ and $\boldsymbol{\beta}_0$ can be readily conducted via the empirical distribution of the smoothed bootstrap estimates $\mathcal{L}_{T,B}$.

REMARK 5.1. We exploit the parametric regression model in the bootstrap resampling, under which the mixing-dependent structure of the observed data does not show up in the asymptotic distributions as shown in Theorem 3.3. As discussed in [17], if one has a parametric model that reduces the data generating process to independence sampling, then the parametric bootstrap has properties that are essentially the same as they are when the observations are independently distributed. Therefore, in the resampling procedure, the temporal dependence of the original data is not necessary to be explicitly taken into account.

REMARK 5.2. In addition to the smoothed regression bootstrap, there are two alternative methods which may be applicable for inference of $\boldsymbol{\gamma}_0$. One is the block subsampling method proposed by [26], which was adopted by [13] in the threshold autoregressive models. Another is the nonparametric posterior confident interval approach based on the Markov Chain Monte Carlo (MCMC) adopted by [35] for inference on the two-regime regression model. Whether these methods work for the current four-regime segmented regression with fixed boundary effects and dependent data are interesting future research topics.

6. Degenerated models and model selection. Model (2.1) assumes that there are four segments divided by two boundary hyperplanes where the adjacent regimes have distinct regression coefficients. However, it is possible that the underlying regimes are degenerated with less than four regimes. In this section, we show that the LS estimator (3.2) attains desirable convergence properties even in the degenerated cases, and propose a model selection method for choosing the underlying model.

Given the data sample $\{(Y_t, \boldsymbol{X}_t, \boldsymbol{Z}_{1,t}, \boldsymbol{Z}_{2,t})\}_{t=1}^T$ for $\boldsymbol{Z}_{1,t} \in \mathcal{Z}_1$ and $\boldsymbol{Z}_{2,t} \in \mathcal{Z}_2$, there are five possible degenerated models as follows in addition to the four regime model (2.1).

(a.1). Three-regime model with non-intersected splitting hyperplanes:

$$Y_t = \sum_{k=1}^{3} \boldsymbol{X}_t^{\intercal} \boldsymbol{\beta}_{k0} \mathbb{1}\left\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma}_0)\right\} + \varepsilon_t, \tag{6.1}$$

where the two hyperplanes H_1 and H_2 have no intersection on $\mathcal{Z}_1 \times \mathcal{Z}_2$. Without loss of generality, we suppose that $\boldsymbol{z}_1^{\mathrm{T}}\boldsymbol{\gamma}_{10} \leq \boldsymbol{z}_2^{\mathrm{T}}\boldsymbol{\gamma}_{20}$ for all $(\boldsymbol{z}_1, \boldsymbol{z}_2) \in (\mathcal{Z}_1 \times \mathcal{Z}_2)$. Then, $R_1(\boldsymbol{\gamma}_0) = \{\boldsymbol{z} : \boldsymbol{z}_1^{\mathrm{T}}\boldsymbol{\gamma}_{10} > 0\}, R_2(\boldsymbol{\gamma}_0) = \{\boldsymbol{z} : \boldsymbol{z}_1^{\mathrm{T}}\boldsymbol{\gamma}_{10} \leq 0, \boldsymbol{z}_2^{\mathrm{T}}\boldsymbol{\gamma}_{20} > 0\}$ and $R_3(\boldsymbol{\gamma}_0) = \{\boldsymbol{z} : \boldsymbol{z}_2^{\mathrm{T}}\boldsymbol{\gamma}_{20} \leq 0\}$. The conventional multi-threshold models (e.g., [12] and [23]) correspond to this case.

(a.2). Three-regime regression model with intersected splitting hyperplanes:

$$Y_t = \sum_{k=1}^{3} \boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\beta}_{k0}\mathbb{1}(\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma}_0) + \varepsilon_t, \tag{6.2}$$

where $R_1(\boldsymbol{\gamma}_0) = \{\boldsymbol{z} : \boldsymbol{z}_i^{\mathrm{T}}\boldsymbol{\gamma}_{j,0} > 0, \boldsymbol{z}_j^{\mathrm{T}}\boldsymbol{\gamma}_{j,0} > 0\}, R_2(\boldsymbol{\gamma}_0) = \{\boldsymbol{z} : \boldsymbol{z}_i^{\mathrm{T}}\boldsymbol{\gamma}_{j,0} > 0, \boldsymbol{z}_j^{\mathrm{T}}\boldsymbol{\gamma}_{j,0} \leq 0\}$ and $R_3(\boldsymbol{\gamma}_0) = \{\boldsymbol{z} : \boldsymbol{z}_j^{\mathrm{T}}\boldsymbol{\gamma}_{j,0} \leq 0\}$ for $i \neq j \in \{1, 2\}$. Geometrically, one side of the hyperplane $H_j : \boldsymbol{z}_j^{\mathrm{T}}\boldsymbol{\gamma}_{j,0} = 0$ is split by $H_i : \boldsymbol{z}_i^{\mathrm{T}}\boldsymbol{\gamma}_{i,0} = 0$ that does not extend to the other side of H_j.

(b.1). Two-regime regression model with one splitting hyperplane:

$$Y_t = \sum_{k=1}^{2} \boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\beta}_{k0}\mathbb{1}\left\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma}_0)\right\} + \varepsilon_t, \tag{6.3}$$

where $(\boldsymbol{z}, \boldsymbol{\gamma}_0)$ is either $(\boldsymbol{z}_1, \boldsymbol{\gamma}_{10})$ or $(\boldsymbol{z}_2, \boldsymbol{\gamma}_{20})$ and $R_1(\boldsymbol{\gamma}_0) = \{\boldsymbol{z} : \boldsymbol{z}^{\mathrm{T}}\boldsymbol{\gamma}_0 > 0\}$ and $R_2(\boldsymbol{\gamma}_0) = \{\boldsymbol{z} : \boldsymbol{z}^{\mathrm{T}}\boldsymbol{\gamma}_0 \leq 0\}$, which are the same as the two-regime models of [22] and [35].

(b.2). Two-regime regression model with two splitting hyperplanes:

$$Y_t = \sum_{k=1}^{2} \boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\beta}_{k0}\mathbb{1}\left\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma}_0)\right\} + \varepsilon_t, \tag{6.4}$$

where $R_1(\boldsymbol{\gamma}_0) = \{\boldsymbol{z} : \boldsymbol{z}_1^{\mathrm{T}}\boldsymbol{\gamma}_{10} > 0, \boldsymbol{z}_2^{\mathrm{T}}\boldsymbol{\gamma}_{20} > 0\}$ and $R_2\{\boldsymbol{\gamma}_0\} = \mathcal{Z}_1 \times \mathcal{Z}_2 \setminus R_1(\boldsymbol{\gamma}_0)$.

(c). Global linear model:

$$Y_t = \boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\beta}_0 + \varepsilon_t, \tag{6.5}$$

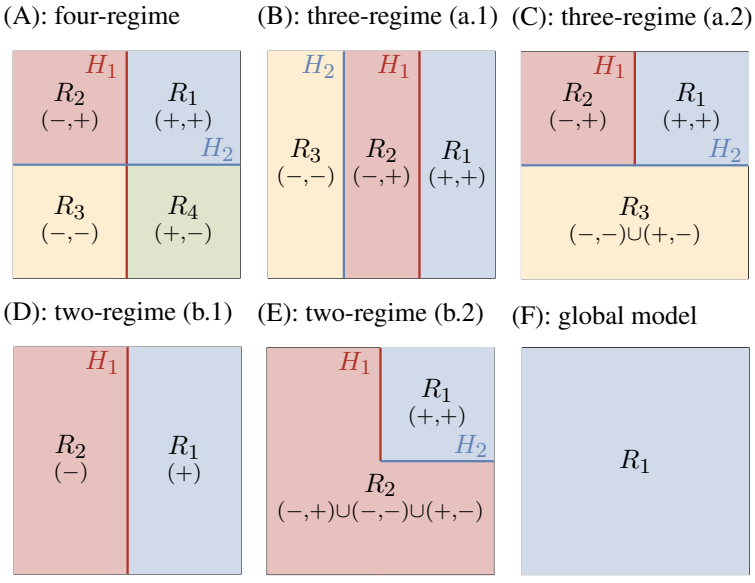


Fig 1: Illustrations of segmented models with no more than four regimes. The signs of $(\boldsymbol{z}_1^{\mathrm{T}}\boldsymbol{\gamma}_1, \boldsymbol{z}_2^{\mathrm{T}}\boldsymbol{\gamma}_2)$ for each region are indicated below the region names.

Figure 1 illustrates the segmented models with no more than four regimes, which can be expressed in a unified form

$$Y_t = \sum_{k=1}^{K_0} \boldsymbol{X}_t^{\mathrm{T}} \boldsymbol{\beta}_{k0} \mathbb{1}\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma}_0)\} + \varepsilon_t, \tag{6.6}$$

where the number of regimes $1 \le K_0 < 4$ and the number of splitting hyperplanes $L_0 \le 2$. In particular, $R_k(\boldsymbol{\gamma}_0) = \mathcal{Z}_1 \times \mathcal{Z}_2$ for the global linear model ($K_0 = 1$), the splitting coefficient $\boldsymbol{\gamma}_0 = \boldsymbol{\gamma}_{10}$ or $\boldsymbol{\gamma}_{20}$ when $L_0 = 1$, and $\boldsymbol{\gamma}_0 = (\boldsymbol{\gamma}_{10}^{\mathrm{T}}, \boldsymbol{\gamma}_{20}^{\mathrm{T}})^{\mathrm{T}}$ when $L_0 = 2$.

Let $\widehat{\mathcal{B}} = \{\widehat{\boldsymbol{\beta}}_k\}_{k=1}^4$ and $\widehat{\mathcal{G}} = \{\widehat{\boldsymbol{\gamma}}_j\}_{j=1}^2$ be the LS estimators for the regression and the boundary coefficients, respectively, obtained under the four-regime regression model (3.2). To measure the estimation accuracy of the four-regime algorithms for less than four regime models, we need a distance of the true parameters of possibly degenerated models to the set of the LS estimates under the four-regime model. To this end, we define a distance between a vector $\boldsymbol{v}$ and a set of vectors $\widehat{\mathcal{V}} = \{\hat{\boldsymbol{v}}_j\}_{j=1}^J$ as $d(\boldsymbol{v}, \widehat{\mathcal{V}}) = \min_j \|\boldsymbol{v} - \hat{\boldsymbol{v}}_j\|_2$. The following theorem establishes the convergence of the LS estimators to the underlying parameters by showing that the distance of the true parameters of the degenerated models to the set of the LSEs under the four-regime model convergences to zero.

THEOREM 6.1. *For Model (6.6) with K_0 regimes and L_0 splitting hyperplanes, where $1 \le K_0 < 4$ and $0 \le L_0 \le 2$, under Assumption 1 and Assumptions S2-S4 in the SM ([33]), which adapt Assumptions 3–4 to the degenerate model settings, then for each $\boldsymbol{\beta}_{k0}$ with $1 \le k \le K_0$, $d(\boldsymbol{\beta}_{k0}, \widehat{\mathcal{B}}) = O_p(1/\sqrt{T})$. If $L_0 = 1$, then $d(\boldsymbol{\gamma}_0, \widehat{\mathcal{G}}) = O_p(1/T)$. If $L_0 = 2$, then $d(\boldsymbol{\gamma}_{i0}, \widehat{\mathcal{G}}) = O_p(1/T)$ for each $i = 1$ and 2. Moreover, for any of the degenerated models with $K_0 < 4$ regimes, there exists an index set $\mathcal{Q}_k \subset \{1, \cdots, 4\}$ such that $\mathbb{P}\{\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0) \triangle \cup_{i \in \mathcal{Q}_k} R_i(\widehat{\boldsymbol{\gamma}})\} = O(1/T)$ for each $1 \le k \le K_0$.*

The theorem shows that under each of the degenerated models, the estimated boundaries and the regression coefficients obtained under (3.2) of the four-regime model are consistent to the true parameters in the sense of the diminishing distance between the true parameters and the sets of the estimates. A remaining issue is to identify the true number of regimes so that more precise segmented regression can be conducted. In the following, we introduce a model selection procedure to attain the purpose.

The last part of Theorem 6.1 suggests that each true regime $R_k(\boldsymbol{\gamma}_0)$ can either be consistently estimated by some $R_i(\widehat{\boldsymbol{\gamma}})$ if $|\mathcal{Q}_k| = 1$, which occurs when $R_k(\boldsymbol{\gamma}_0)$ has two boundaries, such as the first two regimes in Figure 1 (C), or there are some redundant estimated segments in $R_k(\boldsymbol{\gamma}_0)$, which happens if $R_k(\boldsymbol{\gamma}_0)$ has a single boundary while an unnecessary estimated hyperplane splits through $R_k(\boldsymbol{\gamma}_0)$. If the latter case is true, then $|\mathcal{Q}_k| > 1$ and there exist two adjacent estimated regimes $R_i(\widehat{\boldsymbol{\gamma}})$ and $R_h(\widehat{\boldsymbol{\gamma}})$ with $i, h \in \mathcal{Q}_k$, whose corresponding $\widehat{\boldsymbol{\beta}}_i$ and $\widehat{\boldsymbol{\beta}}_h$ both consistently estimate $\boldsymbol{\beta}_{k0}$. Under such a case, merging $R_i(\widehat{\boldsymbol{\gamma}})$ with $R_h(\widehat{\boldsymbol{\gamma}})$ as one regression regime will asymptotically not lead to an increased sum of squared residuals (SSR). Otherwise, if the regression models on $R_i(\widehat{\boldsymbol{\gamma}})$ and $R_h(\widehat{\boldsymbol{\gamma}})$ are distinct, then merging these two regimes will deteriorate the fitting performance. Such a property hints that the true model with $K_0 < 4$ can be selected via a backward elimination procedure.

Starting from the estimated four-regime model, we try recursively finding the best pairs of adjacent regimes to be merged, under a criterion that the merging leads to the minimal increase in the fitting errors, as defined in (6.7) below. Via conducting the optimal regime merging recursively, we obtain four candidate regression models with the number of regimes from $K = 4$ to $K = 1$. In the second step, the optimal number of regimes K is selected

based on a criterion function (6.8) that combines a goodness-of-fit measure and a penalty for over-segmentation.

For the initial model with four regimes, define

$$S_T(4) = \sum_{t=1}^{T}[Y_t - \sum_{k=1}^{K} \boldsymbol{X}_t^{\top}\widehat{\boldsymbol{\beta}}_k^{(4)}\mathbb{1}\{\boldsymbol{Z}_t \in \widehat{R}_k^{(4)}\}]^2$$

to be the sum of square residual (SSR) of the estimated four-regime model. For $K = 4, 3, 2$, recursively define

$$\begin{aligned} &D_T^{(K)}(i,h) \\ &= \min_{\boldsymbol{\beta}\in\mathcal{B}} \sum_{t=1}^{T}[Y_t - \boldsymbol{X}_t^{\top}\boldsymbol{\beta}\mathbb{1}\{\boldsymbol{Z}_t \in \widehat{R}_i^{(K)} \cup \widehat{R}_h^{(K)}\}]^2 - \sum_{t=1}^{T}[Y_t - \sum_{k=i,h} \boldsymbol{X}_t^{\top}\widehat{\boldsymbol{\beta}}_k^{(K)}\mathbb{1}\{\boldsymbol{Z}_t \in \widehat{R}_k^{(K)}\}]^2 \end{aligned}$$

to be the increment in the SSR after merging $\widehat{R}_i^{(K)}$ and $\widehat{R}_h^{(K)}$. Let $\mathcal{A}_\kappa$ be the pair of indices for the adjacent segments of $\{\widehat{R}_k^{(K)}\}$. We merge the segments $\widehat{R}_{\widehat{i}}^{(K)}$ and $\widehat{R}_{\widehat{h}}^{(K)}$ if

$$(\widehat{i},\widehat{h}) = \arg\min_{(i,h)\in\mathcal{A}_K} D_T^{(K)}(i,h), \tag{6.7}$$

followed by labeling the merged region and the remaining regions as $\{\widehat{R}_k^{(K-1)}\}_{k=1}^{K-1}$, and we denote the estimated regression coefficients to these $K-1$ regimes by $\{\widehat{\boldsymbol{\beta}}_k^{(K-1)}\}_{k=1}^{K-1}$. Then, define the SSR of the $(K-1)$-segment submodel as

$$S_T(K-1) = S_T(K) + D_T^{(K)}(\widehat{i},\widehat{h}).$$

After obtaining the $S_T(K)$ for $K = 2, 3, 4$, we select the number of segments $\widehat{K}$ as

$$\widehat{K} = \arg\min_{1\le K\le 4}\{\log(\frac{S_T(K)}{T}) + \frac{\lambda_T}{T}K\} \tag{6.8}$$

and output the estimated regimes and regression coefficients accordingly. The following theorem shows that the above selection algorithm has the model selection consistency.

THEOREM 6.2. *Under the assumptions of Theorem 6.1, and $\lambda_T \to \infty, \lambda_T/T \to 0$ as $T \to \infty$, then $\widehat{K}$ selected in (6.8) satisfies $\mathbb{P}(\widehat{K} = K_0) \to 1$ as $T \to \infty$. In addition, $\mathbb{P}\{\widehat{R}_k^{(\widehat{K})} \triangle R_k(\boldsymbol{\gamma}_0)\} = O(1/T)$ and $\|\widehat{\boldsymbol{\beta}}_k^{(\widehat{K})} - \boldsymbol{\beta}_{k0}\| = O_p(1/\sqrt{T})$ for any $k \in \{1, \cdots, K_0\}$.*

Theorem 6.2 indicates that with the probability approaching 1, the selected number of regimes $\widehat{K}$ coincides with the true number K_0, and as a by-product, the corresponding estimated regimes and the regression coefficients converge to their underlying counterparts. If the regularization parameter is chosen as $\lambda_T = \log T$, the (6.8) corresponds to the Bayesian information criterion (BIC) [30].

REMARK 6.1. There are two existing approaches for carrying out the model selection for the segmented models. One is by conducting pairwise linearity tests. Specifically, for each adjacent regimes $R_i(\widehat{\boldsymbol{\gamma}})$ and $R_h(\widehat{\boldsymbol{\gamma}})$ under the four-regime model, one can test for the hypothesis $H_0 : \boldsymbol{\beta}_{i0} = \boldsymbol{\beta}_{h0}$ via two-regime linearity tests, such as the score-type test of [35]. However, implementing such tests are computationally demanding, as the test statistics have to be formulated via supremum or averaging over $\boldsymbol{\gamma} \in \Gamma$, as $\boldsymbol{\gamma}$ is not identifiable under the null hypothesis of no splitting within $R_i(\widehat{\boldsymbol{\gamma}}) \cup R_h(\widehat{\boldsymbol{\gamma}})$, which is known as the Davis problem [9].

The other is the forward sequential fitting procedure for model selection of multi-threshold regression models [12], which requires optimization for the splitting (boundary) coefficients in each step. Compared with these two methods, the proposed model selection method has two advantages. One is that it has quite readily computation without having to do the bootstrap for the model selection; and the other is that we only need to estimate the splitting coefficients for the initial four-segment model once and for all, as the submodels with fewer regimes are selected via (6.7) without the need to conduct non-convex optimization as in the forward sequential fitting procedure.

7. Simulation Study. In this section, we present results from simulation experiments designed to investigate the performance of the proposed estimation and inference procedures for the four-regime and the degenerated less than four regime models.

7.1. *Estimation under the four-regime model.* We first conducted simulations under the four-regime model (2.1) such that the sample was generated according to

$$Y_t = \sum_{k=1}^{4} \boldsymbol{X}_t^{\mathrm{T}} \boldsymbol{\beta}_{k0} \mathbb{1}_k(\boldsymbol{Z}_{1,t}^{\mathrm{T}} \boldsymbol{\gamma}_{10}, \boldsymbol{Z}_{2,t}^{\mathrm{T}} \boldsymbol{\gamma}_{20}) + \varepsilon_t, \quad t = 1, \cdots, T, \tag{7.1}$$

where $\boldsymbol{X}_t = (\tilde{\boldsymbol{X}}_t^{\mathrm{T}}, 1)^{\mathrm{T}}$ with $\tilde{\boldsymbol{X}}_t = (X_{1,t}, X_{2,t}, X_{3,t})^{\mathrm{T}}$ and $\boldsymbol{Z}_{j,t} = (\tilde{\boldsymbol{Z}}_{j,t}^{\mathrm{T}}, 1)^{\mathrm{T}}$ with $\tilde{\boldsymbol{Z}}_{j,t} = (Z_{j,1,t}, Z_{j,2,t})^{\mathrm{T}}$ for $j = 1, 2$. The noises were generated as $\varepsilon_t = \sigma(\boldsymbol{X}_t, \boldsymbol{Z}_t) e_t$ with $\sigma(\boldsymbol{X}_t, \boldsymbol{Z}_t) = 1 + 0.1 X_{1,t}^2 + 0.1 Z_{1,1,t}^2$ and $\{e_t\}_{t=1}^T$ being generated independently from the standard normal distribution and independent of $\{\boldsymbol{X}_t, \boldsymbol{Z}_t\}_{t=1}^T$. The regression coefficients of the four regimes were $\boldsymbol{\beta}_{10} = (1, 1, 1, 1)^{\mathrm{T}}, \boldsymbol{\beta}_{20} = (-3, -2, -1, 0), \boldsymbol{\beta}_{30} = (0, 1, 3, -1)^{\mathrm{T}}$ and $\boldsymbol{\beta}_{40} = (2, -1, 0, 2)^{\mathrm{T}}$, and the two boundary coefficients $\boldsymbol{\gamma}_{10} = (1, -1, 0)^{\mathrm{T}}$ and $\boldsymbol{\gamma}_{20} = (1, 1, 0)^{\mathrm{T}}$, respectively.

We considered three settings for $\boldsymbol{X}_t$ and $\boldsymbol{Z}_{j,t}$: independence, dependence with autoregressive (AR) and moving average (MA) models, respectively. Let $\boldsymbol{V}_t = (\tilde{\boldsymbol{X}}_t^{\mathrm{T}}, \tilde{\boldsymbol{Z}}_{1,t}^{\mathrm{T}}, \tilde{\boldsymbol{Z}}_{2,t}^{\mathrm{T}})^{\mathrm{T}}$. For the independence setting, we generated $\{\boldsymbol{V}_t\}_{t=1}^T \overset{\text{i.i.d.}}{\sim} \boldsymbol{N}(\boldsymbol{0}, \Sigma_V)$, where $\Sigma_V = (\sigma_{ij})_{i,j=1,\cdots,7}$ with $\sigma_{ii} = 1$ and $\sigma_{ij} = 0.1$ if $i \neq j$. For the AR dependence, $\boldsymbol{V}_t = \psi \boldsymbol{V}_{t-1} + \boldsymbol{u}_t$, where $\{\boldsymbol{u}_t\}_{t=1}^T \overset{\text{i.i.d.}}{\sim} \boldsymbol{N}(\boldsymbol{0}, \Sigma_V)$ and the dependence level $\psi \in \{0.2, 0.4, 0.8\}$. For the MA scenario, we generated $\boldsymbol{V}_t = \psi \boldsymbol{u}_{t-1} + \boldsymbol{u}_t$, where $\{\boldsymbol{u}_t\}_{t=1}^T \overset{\text{i.i.d.}}{\sim} \boldsymbol{N}(\boldsymbol{0}, \Sigma_V)$ and ψ took values in $\{0.2, 0.4, 0.8\}$, respectively. The simulation experimented with four sample sizes: $\{200, 400, 800, 1600\}$, and the experiments were repeated 500 times for each sample size and dependence setting.

Table 1 reports the average L_2 estimation errors under the three temporal settings (independence, AR(1) and MA(1)) and different dependence levels ($\psi = 0.2, 0.4, 0.8$) for $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, respectively. It suggests that under the three dependence settings the estimation errors of $\widehat{\boldsymbol{\gamma}}$ and $\widehat{\boldsymbol{\beta}}$ both decreased as the sample size T was increased, indicating the convergence of the estimation in both the regression and the splitting boundary coefficients. The table also suggests that the magnitudes of the estimation errors were comparable across the three temporal settings with different dependence levels, which support the result of Theorem 3.3 that the temporal dependence in $\{\boldsymbol{X}_t, \boldsymbol{Z}_{1,t}, \boldsymbol{Z}_{2,t}\}_t^T$ does not have leading order effects on the asymptotic variance of $\widehat{\boldsymbol{\beta}}$. Moreover, Table 1 shows that the simulated averages of $\|\boldsymbol{\gamma}_0 - \widehat{\boldsymbol{\gamma}}\|_2$ were approximately halved once the sample size was doubled, while the reduction in $\|\boldsymbol{\beta}_0 - \widehat{\boldsymbol{\beta}}\|_2$ was much slower, confirming the faster convergence rates of $\widehat{\gamma}$.

TABLE 1
Empirical average estimation errors $\|\gamma_0 - \widehat{\gamma}\|_2$ and $\|\boldsymbol{\beta}_0 - \widehat{\boldsymbol{\beta}}\|_2$ (multiplied by 10), under the independence (IND), auto-regressive (AR) and moving average (MA) settings with different dependence level ψ for $\{\boldsymbol{X}_t, \boldsymbol{Z}_{1,t}, \boldsymbol{Z}_{2,t}\}_{t=1}^T$. The numbers inside the parentheses are the standard errors of the simulated averages.

T	IND		AR						MA					
	$\psi=0$		$\psi=0.2$		$\psi=0.4$		$\psi=0.8$		$\psi=0.2$		$\psi=0.4$		$\psi=0.8$	
	$\widehat{\gamma}$	$\widehat{\boldsymbol{\beta}}$	$\widehat{\gamma}$	$\widehat{\boldsymbol{\beta}}$	$\widehat{\gamma}$	$\widehat{\boldsymbol{\beta}}$	$\widehat{\gamma}$	$\widehat{\boldsymbol{\beta}}$	$\widehat{\gamma}$	$\widehat{\boldsymbol{\beta}}$	$\widehat{\gamma}$	$\widehat{\boldsymbol{\beta}}$	$\widehat{\gamma}$	$\widehat{\boldsymbol{\beta}}$
200	0.94 (0.59)	6.68 (1.7)	0.92 (0.58)	6.66 (1.68)	0.88 (0.6)	6.43 (1.56)	0.88 (0.61)	5.9 (2.24)	0.93 (0.56)	6.63 (1.63)	0.9 (0.54)	6.49 (1.66)	0.85 (0.52)	6.14 (1.8)
400	0.45 (0.3)	4.55 (1.1)	0.45 (0.3)	4.55 (1.11)	0.45 (0.27)	4.4 (1.17)	0.43 (0.29)	3.98 (1.53)	0.44 (0.28)	4.46 (1)	0.43 (0.33)	4.38 (1.07)	0.43 (0.28)	4.06 (1.21)
800	0.25 (0.16)	3.11 (0.66)	0.24 (0.15)	3.09 (0.66)	0.22 (0.14)	2.97 (0.66)	0.22 (0.14)	2.64 (0.96)	0.23 (0.14)	3.11 (0.66)	0.25 (0.16)	3.03 (0.65)	0.22 (0.15)	2.81 (0.72)
1600	0.11 (0.07)	2.2 (0.46)	0.11 (0.07)	2.18 (0.47)	0.12 (0.08)	2.11 (0.5)	0.11 (0.07)	1.88 (0.77)	0.11 (0.07)	2.17 (0.45)	0.11 (0.07)	2.11 (0.47)	0.11 (0.07)	1.97 (0.54)

7.2. *Estimation under models with less than four regimes.* We next investigated the performances of the proposed estimation based on the four-regime model when the underlying model was degenerated with less than four regimes. The data generating process for $\{\boldsymbol{X}_t, \boldsymbol{Z}_{1,t}, \boldsymbol{Z}_{2,t}, \varepsilon_t\}_{t=1}^T$ was largely the independence setting used in Section 7.1. For the three-regime model (6.1) with non-intersected splitting hyperplanes, we let $\gamma_{10} = (1,0,-1)^{\mathrm{T}}, \gamma_{20} = (1,0,1)^{\mathrm{T}}$ and $\boldsymbol{\beta}_{10} = (1,1,1,1)^{\mathrm{T}}, \boldsymbol{\beta}_{20} = (-3,-2,-1,0)^{\mathrm{T}}, \boldsymbol{\beta}_{30} = (0,1,3,-1)^{\mathrm{T}}$. For the three-regime model (6.2) with intersected splitting hyperplanes, we let $\gamma_{10} = (1,1,0)^{\mathrm{T}}, \gamma_{20} = (1,-1,0)^{\mathrm{T}}$ while H_{10} does not extend to the positive side of H_{20}, and $\{\boldsymbol{\beta}_{k0}\}_{k=1}^3$ were the same as above. The parameters for the two-regime model (6.3) with one splitting hyperplane were set as $\gamma_0 = (1,1,0)^{\mathrm{T}}$, $\boldsymbol{\beta}_{10} = (1,1,1,1)^{\mathrm{T}}$ and $\boldsymbol{\beta}_{20} = (-3,-2,-1,0)^{\mathrm{T}}$. For the two-regime model (6.4) with two splitting hyperplanes, we set the splitting coefficients as the same as the four-regime model (7.1), and $R_1(\gamma_0) = \{\boldsymbol{z} : \boldsymbol{z}_1^{\mathrm{T}}\gamma_{10} > 0, \boldsymbol{z}_2^{\mathrm{T}}\gamma_{20} > 0\}$ and $R_2(\gamma_0) = \mathcal{Z}_1 \times \mathcal{Z}_2 \setminus R_1(\gamma_0)$, where the regression coefficients are $\boldsymbol{\beta}_{10} = (1,1,1,1)^{\mathrm{T}}$ and $\boldsymbol{\beta}_{20} = (-3,-2,-1,0)^{\mathrm{T}}$, respectively. Finally, the regression coefficients for the global linear model (6.5) were $\boldsymbol{\beta}_0 = (1,1,1,1)^{\mathrm{T}}$.

The simulation results are reported in Tables S2 of Section H.2 in the SM ([33]). They show that for all the models with less than four regimes, the empirical averages of $\sum_i d(\gamma_{i0}, \widehat{\mathcal{G}})$ and $\sum_k d(\boldsymbol{\beta}_{k0}, \widehat{\mathcal{B}})$ all diminished to 0 at similar rates as those in Table 1, where $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{B}}$ are the sets of estimators obtained under the four-regime model for the splitting and regression coefficients, respectively. These confirmed the results in Theorem 6.1. In addition, to evaluate the cost of not knowing the number of the underlying regimes, we also estimated γ_0 and $\boldsymbol{\beta}_0$ in the oracle setting, in which the true model forms were known. It was found that estimation errors of γ_0 under the four-regime model fitting were about the same as that obtained under the oracle models, which was because the four-regime estimator can efficiently use the data points located near the underlying boundaries as the oracle estimators did. Moreover, as shown in Figures S2 and S3 of the SM ([33]), if the estimated four-regime model produced redundant segments within a true regime, then the discrepancy between the estimated regression coefficients on these redundant segments converged to 0, which verified the idea used in the optimal merger strategy for the backward elimination procedure in the model selection.

7.3. *Model selection.* We then conducted simulation experiments to examine the performance of the proposed model selection method in Section 6. We considered the true number

of regimes ranging from $K_0 = 4$ to $K_0 = 1$, where the parameters for the model with $K_0 = 4$ were the same as Model (7.1) and those for $K_0 = 3$ and $K_0 = 2$ were Model 6.1 and Model 6.3, respectively, in Section 7.2. More simulation results for Model (6.2) and Model (6.5) ($K_0 = 1$) were reported in Table S3 of the SM.

TABLE 2

Empirical model selection results under 500 *replications. The performances were evaluated by the average estimated number of regimes $\widehat{K}$, the discrepancy between the true regimes and the estimated regimes $D(\mathcal{R}, \widehat{\mathcal{R}})$ and the L_2 estimation error of regression coefficients $D(\mathcal{B}, \widehat{\mathcal{B}})$. The penalty parameter λ_T was chosen in $\{5, 5\log(T), 5\log^2(T)\}$. The numbers inside the parentheses are the standard errors of the simulated averages.*

Model	T	$\lambda_T = 5$			$\lambda_T = 5\log(T)$			$\lambda_T = 5\log^2(T)$		
		$\widehat{K}$	$D(\mathcal{R}, \widehat{\mathcal{R}})$	$D(\mathcal{B}, \widehat{\mathcal{B}})$	$\widehat{K}$	$D(\mathcal{R}, \widehat{\mathcal{R}})$	$D(\mathcal{B}, \widehat{\mathcal{B}})$	$\widehat{K}$	$D(\mathcal{R}, \widehat{\mathcal{R}})$	$D(\mathcal{B}, \widehat{\mathcal{B}})$
Model (2.1) ($K_0 = 4$)	200	4.00 (0.00)	0.03 (0.02)	0.61 (0.12)	3.99 (0.08)	0.03 (0.04)	0.62 (0.16)	2.78 (0.87)	0.87 (0.91)	2.24 (1.05)
	400	4.00 (0.00)	0.01 (0.01)	0.41 (0.08)	4.00 (0.00)	0.01 (0.01)	0.41 (0.08)	3.92 (0.27)	0.05 (0.13)	0.53 (0.43)
	800	4.00 (0.00)	0.01 (0.00)	0.29 (0.05)	4.00 (0.00)	0.01 (0.00)	0.29 (0.05)	4.00 (0.00)	0.01 (0.00)	0.29 (0.05)
	1600	4.00 (0.00)	0.00 (0.00)	0.20 (0.04)	4.00 (0.00)	0.00 (0.00)	0.20 (0.04)	4.00 (0.00)	0.00 (0.00)	0.20 (0.04)
Model (6.1) ($K_0 = 3$)	200	3.44 (0.50)	0.12 (0.11)	0.50 (0.11)	3.00 (0.00)	0.02 (0.02)	0.48 (0.11)	2.85 (0.38)	0.13 (0.30)	0.75 (0.69)
	400	3.39 (0.49)	0.10 (0.11)	0.34 (0.07)	3.00 (0.00)	0.01 (0.01)	0.33 (0.07)	3.00 (0.00)	0.01 (0.01)	0.33 (0.07)
	800	3.33 (0.47)	0.08 (0.11)	0.23 (0.05)	3.00 (0.00)	0.01 (0.00)	0.22 (0.05)	3.00 (0.00)	0.01 (0.00)	0.22 (0.05)
	1600	3.33 (0.47)	0.08 (0.11)	0.16 (0.03)	3.00 (0.00)	0.00 (0.00)	0.16 (0.03)	3.00 (0.00)	0.00 (0.00)	0.16 (0.03)
Model (6.3) ($K_0 = 2$)	200	3.38 (0.59)	0.14 (0.11)	0.35 (0.10)	2.03 (0.17)	0.01 (0.01)	0.30 (0.08)	2.00 (0.00)	0.01 (0.01)	0.30 (0.08)
	400	3.54 (0.51)	0.13 (0.11)	0.24 (0.07)	2.01 (0.08)	0.01 (0.01)	0.20 (0.05)	2.00 (0.00)	0.01 (0.00)	0.20 (0.05)
	800	3.53 (0.53)	0.12 (0.11)	0.16 (0.04)	2.00 (0.06)	0.00 (0.00)	0.14 (0.04)	2.00 (0.00)	0.00 (0.00)	0.14 (0.04)
	1600	3.50 (0.55)	0.13 (0.12)	0.12 (0.03)	2.00 (0.00)	0.00 (0.00)	0.10 (0.03)	2.00 (0.00)	0.00 (0.00)	0.10 (0.03)

Table 2 reports three model selection performance measures for the simulation, namely (i) the estimated number of regimes $\widehat{K}$, (ii) the discrepancy between the true regimes and the estimated regimes measured by

$$D(\mathcal{R}, \widehat{\mathcal{R}}) = \sum_{k=1}^{K_0} \min_{1 \le h \le \widehat{K}} \left\{ T^{-1} \sum_{t=1}^{T} |\mathbb{1}\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma}_0)\} - \mathbb{1}\{\boldsymbol{Z}_t \in R_h(\widehat{\boldsymbol{\gamma}})\}| \right\},$$

where $\mathcal{R} = \{R_k(\boldsymbol{\gamma})\}_{k=1}^{K_0}$ and $\widehat{\mathcal{R}} = \{R_k(\widehat{\boldsymbol{\gamma}})\}_{k=1}^{\widehat{K}}$, and (iii) the L_2 estimation error of regression coefficients, quantified by $D(\mathcal{B}, \widehat{\mathcal{B}}) = \sum_{k=1}^{K_0} \min_{1 \le h \le \widehat{K}} \|\boldsymbol{\beta}_{k0} - \widehat{\boldsymbol{\beta}}_h\|$. To evaluate the impact of the penalty parameter λ_T in (6.8), we presented the results under three different choices: $\lambda_T = 5, 5\log(T)$ and $5\log^2(T)$.

Table 2 shows that, for the constant penalty $\lambda_T = 5$, although the estimated number of regimes $\widehat{K}$ was consistent under $K_0 = 4$, it tended to select overly segmented models when $K_0 < 4$. Both $\lambda_T = 5\log(T)$ and $5\log^2(T)$ led to consistent estimated $\widehat{K}$ for all models, which confirmed the assertion in Theorem 6.2 that λ_T satisfying $\lambda_T \to \infty$ and $\lambda_T/T \to 0$ leads to model selection consistency. It was also noted that while the last two penalties were consistent, for smaller sample sizes, the selection performance with $\lambda_T = 5\log(T)$ was superior to that with $\lambda_T = 5\log^2(T)$ when $K_0 \geq 3$, while the latter penalty had better selection accuracy when $K \leq 2$. Such a phenomenon may be understood since a larger penalty tends to encourage under-segmentations. In addition, both $D(\mathcal{R}, \widehat{\mathcal{R}})$ and $D(\mathcal{B}, \widehat{\mathcal{B}})$ diminished to 0 when $\widehat{K}$ was correctly selected, indicating that the model specification procedure was able to not only consistently identify K_0, but also led to consistent estimates of regimes and the corresponding regression coefficients, as shown in Theorem 6.2.

7.4. *Smoothed regression bootstrap.* We now report simulation results designed to evaluate the empirical performance of the smoothed regression bootstrap.

The data generating model for $\{Y_t, \boldsymbol{X}_t, \boldsymbol{Z}_{1,t}, \boldsymbol{Z}_{2,t}\}_{t=1}^T$ was the same as the independent setting in Section 7.1, but $(\tilde{\boldsymbol{X}}_t^{\mathrm{T}}, \tilde{\boldsymbol{Z}}_{1,t}^{\mathrm{T}}, \tilde{\boldsymbol{Z}}_{2,t}^{\mathrm{T}})^{\mathrm{T}}$ was truncated over a 7-dimensional region $[-2, 2]^7$ to ensure the distribution of the covariates was compactly supported as required in Assumption 6. The product Gaussian kernel was used as the kernel function with the smoothing bandwidths h_i and $b_i (i = 1, 2)$ for $\tilde{F}_0(\boldsymbol{x}, \boldsymbol{z})$ and $\tilde{\sigma}^2(\boldsymbol{x}, \boldsymbol{z})$ were chosen by the cross-validation method ([10]). As a comparison, we also conducted the wild bootstrap procedure ([24]), which is a commonly used bootstrap method in regression. Different from the smoothed regression bootstrap, the wild bootstrap does not resample the covariates and the resampled residuals $\varepsilon_t^* = d_t^* \widehat{\varepsilon}_t$, where $\widehat{\varepsilon}_t$ was the estimated residual and d_t^* followed a two-point distribution. Both the smoothed regression bootstrap and the wild bootstrap were based on $B = 500$ resamples for each simulation run. As there are infinitely many solutions for $\hat{\boldsymbol{\gamma}}$ from the MIQP algorithm, for each bootstrap resample, we outputted $N = 100$ solutions for the LSE of $\boldsymbol{\gamma}_0$ and used their average as $\widehat{\boldsymbol{\gamma}}_b^{*c}$.

TABLE 3

Empirical coverage probabilities and widths ($\times 100$ in parentheses) of the 95% confidence intervals for five projected parameters $\{\tilde{\boldsymbol{\gamma}}^{\mathrm{T}} \boldsymbol{d}_i\}_{i=1}^5$ obtained with the smoothed regression bootstrap (Smooth) and the wild bootstrap (Wild) based on 500 resamples.

T	$\boldsymbol{d}_1$		$\boldsymbol{d}_2$		$\boldsymbol{d}_3$		$\boldsymbol{d}_4$		$\boldsymbol{d}_5$	
	Smooth	Wild	Smooth	Wild	Smooth	Wild	Smooth	Wild	Smooth	Wild
200	0.92 (6.76)	0.87 (3.57)	0.97 (6.91)	0.87 (3.91)	0.93 (5.78)	0.90 (4.02)	0.93 (6.20)	0.83 (3.44)	0.96 (6.86)	0.86 (3.56)
400	0.95 (3.31)	0.86 (1.69)	0.94 (3.57)	0.83 (1.89)	0.97 (2.56)	0.86 (1.94)	0.94 (3.37)	0.88 (1.73)	0.97 (3.69)	0.85 (1.75)
800	0.93 (1.70)	0.85 (0.83)	0.96 (1.76)	0.87 (0.99)	0.94 (1.68)	0.88 (1.00)	0.96 (1.72)	0.88 (0.86)	0.96 (1.80)	0.87 (0.76)
1600	0.95 (0.81)	0.83 (0.40)	0.94 (0.86)	0.88 (0.51)	0.95 (0.89)	0.90 (0.53)	0.96 (0.85)	0.84 (0.41)	0.94 (0.79)	0.85 (0.42)

To evaluate the quality of the two bootstrap schemes, we constructed 95% confidence intervals (CIs) for $\tilde{\boldsymbol{\gamma}}_0 = (\boldsymbol{\gamma}'_{-1,10}, \boldsymbol{\gamma}'_{-1,20})^{\mathrm{T}} = (-1, 0, 1, 0)^{\mathrm{T}}$ projected on five directions $\{\boldsymbol{d_i}\}_{i=1}^5$ where $\boldsymbol{d_i} = \boldsymbol{e}_i$ for $i = 1, \ldots, 4$ and $\boldsymbol{d}_5 = \sum_{i=1}^4 \boldsymbol{d}_i/2$, and $\boldsymbol{e}_i = (e_{i1}, \cdots, e_{i4})^{\mathrm{T}}$ with $e_{ii} = 1$ and $e_{ij} = 0$ if $j \neq i$. Table 3 reports the coverage probabilities and widths of the nominal 95% CIs

for $\tilde{\gamma}_0^{\mathrm{T}} \boldsymbol{d}_i$ based on the smoothed regression bootstrap and the wild bootstrap, respectively. It is shown that the smoothed regression bootstrap had satisfactory coverage as its empirical coverage levels were quite close to the nominal 95% level under large sample sizes for all the five projection directions. This verified the consistency of the proposed bootstrap procedure in Theorem 5.1. On the other hand, the wild bootstrap had substantial under-coverage, and its coverage was not improved with the increases of the sample sizes. The comparison between the two bootstrap schemes reveals that for the inference of $\boldsymbol{\gamma}_0$, it is crucial to conduct resampling from a smoothed distribution, as advocated in Section 5.

8. Case Study. Air quality is naturally affected by meteorological regimes as the latter defines the atmospheric dispersion conditions. We demonstrate here that the four-regime regression model is well suited for $\mathrm{PM}_{2.5}$ modeling in Beijing.

We considered hourly $\mathrm{PM}_{2.5}$ data from Wanshouxigong site in central Beijing with the meteorological data from the nearest weather observation site being used. The study period was from December 1, 2018 to November 30, 2019, which encompassed four seasons. The meteorological data included the air temperature (TEMP), dew point temperature (DEWP), surface air pressure (PRES), the cumulative wind speed (IWS) at a direction and wind direction (WD). Cumulative rainfall (RAIN) was included in summer, however not in the other three seasons due to a lack of it. The categorical wind direction (WD) took five values: Northwesterly (NW), Northeasterly (NE), Southwesterly (SW), Southeasterly (SE) and calm and variable (CV). We also used the boundary layer height (BLH), which defines the vertical dispersion property, from European Centre for Medium-Range Weather Forecasts (ECMWF).

To investigate the in-sample and out-of-sample performances, the data were divided to the training and testing sets, where the testing sets consisted of the data from the 11-th to the 20-th days of a month and the training sets included the rest of the data in the month. $\mathrm{PM}_{2.5}$ was regressed on covariates TEMP, DEWP, PRES, $\log(\mathrm{BLH})$, IWS, WD as well as the $\mathrm{PM}_{2.5}$ at the previous hour (Lag $\mathrm{PM}_{2.5}$). For the wind direction, NW, NE, SW and SE were set as dummy covariates with the CV as the baseline.

Along with the proposed four-regime model (4-REG), the global linear regression (GLR), the two-regime model (2-REG) [22] and [35], the linear regression tree (LRT) ([37]) and the multivariate adaptive regression splines (MARS) [11] were also considered. For 2-REG and 4-REG, the splitting boundaries were determined by TEMP, DEWP, $\log(\mathrm{BLH})$, IWS, and the four wind directions NE, NW, SE and SW with the coefficients standardized so that the intercept term being 1.

Fig 2: Mean squared errors (MSE) for $\mathrm{PM}_{2.5}$ on the training (red) and testing (green) sets for each season of five models, including global linear regression (GLR), two-regime model (2-REG), four-regime model (4-REG), linear regression tree (LRT) and multivariate adaptive regression splines (MARS), with model ranks (in increasing order of the MSEs) marked on top of the bars.

Figure 2 summarizes the in-sample and out-of-sample MSEs of these models in each season. Within the training sets, LRT or MARS achieved the lowest MSE among the five models with the average rank being 1.75 and 2, respectively. Here, rank 1 indicates the best performance. The average rank of the 4-REG in the training groups was 2.5, while those of the 2-REG and GLR ranked the lowest in all seasons. However, LRT and MARS had the highest prediction MSEs on the testing sets, even worse than the benchmark GLR for all seasons, indicating they were severely over-fitted. The segmented linear models, 4-REG and 2-REG, were the best two in terms of out-of-sample performances, with the 4-REG achieving the lowest predictive errors consistently in all seasons.

The estimated 4-regimes models in the spring, summer and fall seasons all had three regimes, as the fourth estimated regime had zero sample size in the three seasons. A further examination suggested that the two estimated boundaries had no intersections over the sample regions, which corresponded to Model (6.1) and reflected the fact that the proposed LS criterion based on the four-regime model may be able to produce a three-regime model if the latter offers better fit. The winter had four estimated regimes. The estimated regression coefficients and their 95% confidence intervals are given in Figure S4 of the SM ([33]).

TABLE 4

Estimated coefficients of the splitting boundaries and $\cos$ *of the angle* ϕ *between the two boundaries. The coefficients were normalized such that the coefficients of the intercept terms were* 1. *All the covariates were standardized such that their sample means were* 0 *and standard deviations were* 1 *in each season.*

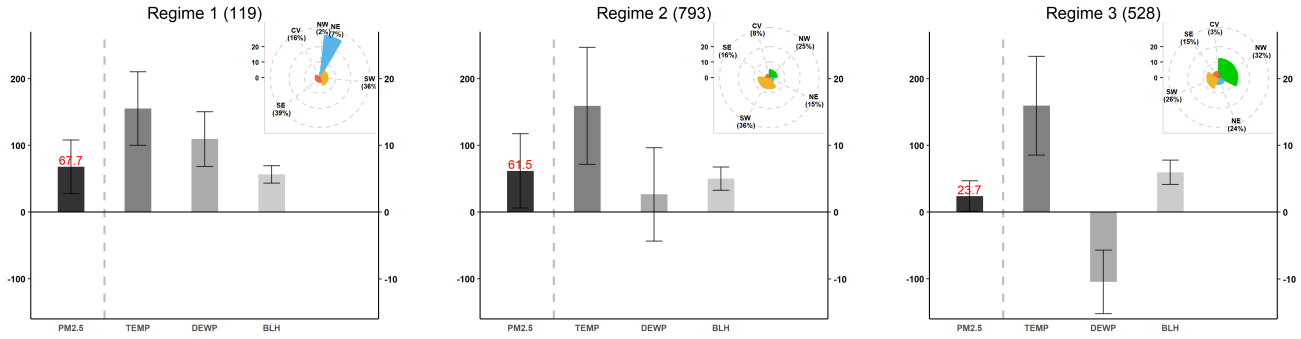
Season	γ	TEMP	DEWP	IWS	log(BLH)	NE	NW	SE	SW	$\cos\phi$
Spring	1	1.3	-2.5	-0.0	-0.4	0.9	0.3	0.1	0.0	0.78
	2	0.4	-0.5	-0.1	-0.1	0.6	0.6	0.1	0.3	
Summer	1	1.0	5.5	-12.9	-0.0	-12.7	-15.0	-8.9	-9.0	0.75
	2	0.4	0.2	-0.2	0.0	-0.7	-0.7	-0.7	-0.7	
Fall	1	0.7	-1.0	0.3	-0.1	0.5	-0.0	0.3	0.0	0.65
	2	-0.5	1.6	-1.0	0.0	0.1	-1.6	-1.3	-0.1	
Winter	1	0.2	-0.5	0.6	-0.2	0.2	0.4	0.4	-0.4	0.45
	2	0.0	-0.6	0.2	-0.4	1.2	1.4	0.3	1.0	

Table 4 reports the estimated coefficients of the two splitting boundaries for each season as well as the cosine of the dihedral angle (denoted as ϕ) between the two boundary hyperplanes. It can be seen that $\cos\phi$ for the first three seasons were relatively larger than that in winter, which explains why the boundary hyperplanes of these three seasons were non-intersected. Table 4 indicates that the DEWP and the wind-related variables were the most influential in determining the slopes of the estimated boundaries due to their absolute coefficient values as the $\boldsymbol{\gamma}$ was normalized. This reveals an attraction of the proposed regime-splitting mechanism in that the splitting boundaries are determined empirically by multivariate covariates, which contrasts to the threshold regression where the boundary variable has to be user-specified.

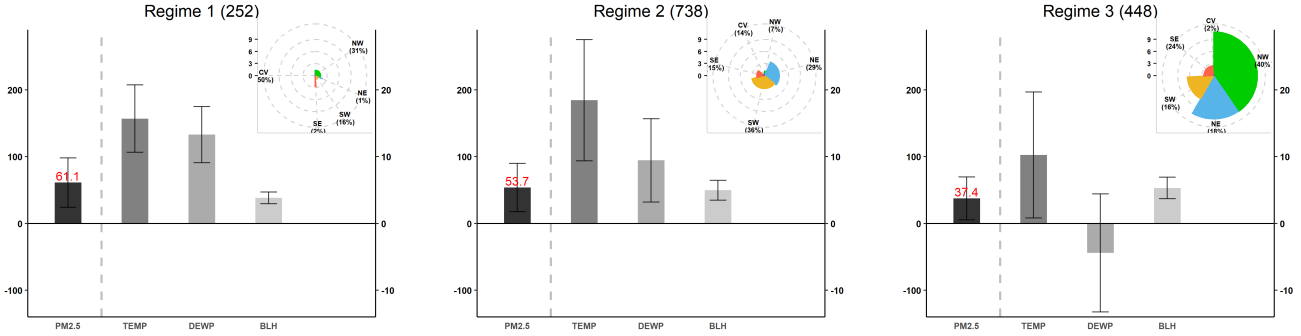
Figure 3 displays summary statistics of $PM_{2.5}$ and the meteorological variables under the three regimes in the spring and fall seasons, as well as the rose plots for the wind directions and the average integrated wind speed (IWS). It shows that the segmented regression picked up three meteorological regimes on $PM_{2.5}$ where Regime 1 corresponded to the pollution state with high DEWP and high proportion of Calm and Variable wind (CV) which are known to encourage the secondary generation of PM2.5 and unfavorable static atmospheric diffusion, Regime 2 was a transitional state between the clean and high pollution states with reduced DEWP and CV, and Regime 3 was a cleaning state dominated by the northerly wind which brought cleaner and cooler air from the north. Results of the other two seasons and analysis are provided in Figure S5 of the SM.

Fig 3: Bar and rose plots for key variables under each estimated regimes in spring and fall 2019. The height of the bars indicate the sample means with imposed line segments indicating twice of the sample deviations above and below the means. The rose plots display the distribution of wind directions (width of angles) and average speed (length of radius). Sample sizes of each regime is reported in the subtitle.

(a) Spring



(b) Fall



9. Discussion. This paper develops a statistical inference approach for four-regimes segmented linear models, which broadens the scope of the two-regime models of [22] and [35], and can attains valid inference for degenerated models with less than four regimes. The proposed segmented model is shown to produce better in-sample and out-sample results for the air quality data in Beijing and produced regime-splitting results which had clear atmospheric physics interpretation.

There are two possible extensions which may be considered in future research. One is to allow endogeneity which may be encountered in economic and social behavior applications. If $\boldsymbol{X}_t$ is endogenous and $\boldsymbol{Z}_t$ is exogenous, $\boldsymbol{\beta}_0$ and $\boldsymbol{\gamma}_0$ can be consistently estimated with instrument variables $\boldsymbol{V}_t$ and the two-stage least squares estimation (2SLS) by first regressing $\boldsymbol{X}_t$ on $\boldsymbol{V}_t$, and then using the fitted $\hat{\boldsymbol{X}}_t$ to substitute $\boldsymbol{X}_t$ in the four-regime model. The LS estimation via the MIQP and the inference methods for the four-regime model presented in this paper is still applicable. However, the 2SLS is no longer working if $\boldsymbol{Z}_t$ is endogenous as discussed in [36], who proposed a conditioning and re-centering approach which might be extended to the four-regime model. Specifically, let $g(\boldsymbol{X}_t, \boldsymbol{Z}_t) = \boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\beta}_{10} + \mathbb{E}(\varepsilon_t|\boldsymbol{X}_t, \boldsymbol{Z}_t)$, $\boldsymbol{\delta}_{k0} = \boldsymbol{\beta}_{k0} - \boldsymbol{\beta}_{10}$ for $k \neq 1$, and $e_t = \varepsilon_t - \mathbb{E}(\varepsilon_t|\boldsymbol{X}_t, \boldsymbol{Z}_t)$, then Model (2.1) can be written as $Y_t = g(\boldsymbol{X}_t, \boldsymbol{Z}_t) + \sum_{k=1}^{3} \boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{k0}\mathbb{1}\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma}_0)\} + e_t$, which is a partially linear segmented

model, where $\boldsymbol{\gamma}_0$ is identifiable without instrument variables. However, the integrated difference kernel estimator used in [36] was designed for univariate threshold, and it is interesting to see how it can be extended to multivariate $\boldsymbol{\gamma}_0$. Alternatively, one may consider estimating $\boldsymbol{\gamma}_0$ via the mixed integer programming with the nonlinear $\mathbb{E}(\varepsilon_t|\boldsymbol{X}_t, \boldsymbol{Z}_t)$ part approximated via sieve functions. How to solve these issues in the context of the four-regime model requires further investigation.

Another extension is for segmented models with $L > 2$ splitting hyperplanes. In general, the L splitting hyperplanes in $\mathbb{R}^d$ can lead to as many as $K_L = \sum_{i=0}^{\min(L,d)} \binom{L}{i}$ segments, as shown in Section G of the SM ([33]). It is clear that the investigations in this study for the two boundary case provide vital understanding to the general cases. For example, if we consider an extension to the case of having three hyperplanes in $\mathbb{R}^d$, we can fit a segmented model with $K = \sum_{i=0}^{\min(3,d)} \binom{3}{i}$ regimes by the least squares estimation, whose criterion function would have the same form as (3.1). The backward selection procedure in Section 6 can be employed to specify the optimal number of regimes, and the smoothed regression bootstrap is still able to facilitate the inference for $\boldsymbol{\gamma}_0$ and $\boldsymbol{\beta}_0$. Furthermore, the proof for the asymptotic distributions of the least squares estimators can be modified to suit the more general segmented models. The main challenge for the general cases is the complicated model form and demanding computation costs caused by the increase of L, requiring efforts in further studies. On the other hand, as K_L grows exponentially with respect to L if $d > L$ and polynomially if $d \leq L$, there would be little need to consider segmented models with large L and d as the nonparametric local models (regression trees, etc) may be better suited.

Acknowledgements. The research was partially supported by National Natural Science Foundation of China grants 12292980, 12292983 and 92358303.

SUPPLEMENTARY MATERIAL

Supplement to "Statistical Inference on Four-Regime Segmented Regression Models" In the supplementary material, we present technical details, proofs and additional results of the simulations and the case study.

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SUPPLEMENT TO "STATISTICAL INFERENCE FOR FOUR-REGIME SEGMENTED REGRESSION MODELS"

BY HAN YAN[1,a] AND SONG XI CHEN[2,b]

[1]*Guanghua School of Management, Peking University,* [a]*hanyan@stu.pku.edu.cn*

[2]*Department of Statistics and Data Science, Tsinghua University,* [b]*sxchen@tsinghua.edu.cn*

The supplementary materials contain additional details, theoretical proofs, and additional results on simulations and case studies of the paper " Statistical Inference for Four-regime Segmented Regression Models".

Organization

Notations. Throughout the supplementary material, we use $c_1, c_2, C_1, C_2, \cdots$ to denote generic finite positive constants, which may differ from line to line. We use $\mathbb{1}(\mathcal{A})$ as the indicator function of an event $\mathcal{A}$. For any vector $\boldsymbol{v} = (v_1, \cdots, v_d)^{\mathrm{T}}$, let $\|\boldsymbol{v}\| = (\sum_{i=1}^{d} v_i^2)^{1/2}$ be its L_2-norm. For any $r > 0$, we define $\mathcal{N}(\boldsymbol{v}_0; r) = \{\boldsymbol{v} : \|\boldsymbol{v} - \boldsymbol{v}_0\| \le r\}$. Denote by $\boldsymbol{v}_{-1}$ as the sub-vector of $\boldsymbol{v}$ excluding its first element, i.e., $\boldsymbol{v}_{-1} = (v_2, \cdots, v_d)^{\mathrm{T}}$. For any two sets A, B, we let $A \setminus B = A \cap B^c$, where B^c is the complement of B, and $A \triangle B = (A \setminus B) \cup (B \setminus A)$. The empirical measure $\mathbb{E}_T(\cdot)$ denotes the sample average of a sequence of random elements with T observations, i.e., $\mathbb{E}_T(\boldsymbol{X}_t) = T^{-1} \sum_{t=1}^{T} \boldsymbol{X}_t$. We also denote $\mathbb{G}_T(\cdot) = \sqrt{T}\{\mathbb{E}_T(\cdot) - \mathbb{E}(\cdot)\}$.

For the four-regime regression model

$$Y_t = \sum_{k=1}^{4} \boldsymbol{X}_t^{\mathrm{T}} \boldsymbol{\beta}_k \mathbb{1}\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma})\} + \varepsilon_t,$$

we define the indicator functions for the t-th observation on the k-th regions as

$$\mathbb{1}_t^{(k)}(\boldsymbol{\gamma}) := \mathbb{1}\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma})\} \text{ for } k \in \{1, \cdots, 4\};$$

and for $l = 1$ and 2, let

$$\mathbb{1}_{l,t}(\boldsymbol{\gamma}) := \mathbb{1}(\boldsymbol{Z}_{l,t}^{\mathrm{T}} \boldsymbol{\gamma} > 0) \quad \text{and} \quad \mathbb{1}_{l,t}(\boldsymbol{\gamma}, \tilde{\boldsymbol{\gamma}}) := \mathbb{1}(\boldsymbol{Z}_{l,t}^{\mathrm{T}} \boldsymbol{\gamma} \le 0 < \boldsymbol{Z}_{l,t}^{\mathrm{T}} \tilde{\boldsymbol{\gamma}}). \tag{1}$$

For each $1 \le k \le 4$, that $\boldsymbol{z} = (\boldsymbol{z}_1, \boldsymbol{z}_2) \in R_k(\boldsymbol{\gamma})$ or not depends on the signs of $\boldsymbol{z}_1^{\mathrm{T}} \boldsymbol{\gamma}_1$ and $\boldsymbol{z}_2^{\mathrm{T}} \boldsymbol{\gamma}_2$. As results, for each $l = 1$ and 2 and $1 \le k \le 4$, we denote

$$s_l^{(k)} = \operatorname{sign}(\boldsymbol{z}_l^{\mathrm{T}} \boldsymbol{\gamma}_l), \text{ for } (\boldsymbol{z}_1, \boldsymbol{z}_2) \in R_k(\boldsymbol{\gamma}), \tag{2}$$

for each $l \in \{1,2\}$ and $k \in \{1,\cdots,4\}$, which is well-defined since any $\boldsymbol{z} \in R_k(\boldsymbol{\gamma})$ has the same $\text{sign}\,(\boldsymbol{z}^{\mathrm{T}}\boldsymbol{\gamma}_l)$. Specifically, in the four-regime model, we have $s_1^{(1)} = s_2^{(1)} = 1; s_1^{(2)} = -1, s_2^{(2)} = 1; s_1^{(3)} = s_2^{(3)} = -1$; and $s_1^{(4)} = 1, s_2^{(4)} = -1$. We now define the pairs of *adjacent* sub-regions. For the l-th splitting hyperplane, we let

$$\mathcal{S}(l) = \left\{(j,k) : s_l^{(j)} \neq s_l^{(k)} \text{ and } s_i^{(j)} = s_i^{(k)} \text{ if } i \neq l\right\}, \tag{3}$$

that is, $\{\boldsymbol{z}_l^{\mathrm{T}}\boldsymbol{\gamma}_l = 0\}$ is the only splitting hyperplane that $R_j(\boldsymbol{\gamma})$ and $R_k(\boldsymbol{\gamma})$ are on opposite directions of it. Specifically, in the four-regime model, $\mathcal{S}(1) = \{(1,2),(3,4),(2,1),(4,3)\}$ and $\mathcal{S}(2) = \{(1,4),(2,3),(4,1),(3,2)\}$. Let

$$m(\boldsymbol{W}_t, \boldsymbol{\theta}) = \{Y_t - \sum_{k=1}^{4} \boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\beta}_k \mathbb{1}_k(\boldsymbol{Z}_{1,t}^{\mathrm{T}}\boldsymbol{\gamma}_1, \boldsymbol{Z}_{2,t}^{\mathrm{T}}\boldsymbol{\gamma}_2)\}^2.$$

We denote by $\mathbb{M}_T(\boldsymbol{\theta}) = \mathbb{E}_T\{m(\boldsymbol{W}_t, \boldsymbol{\theta})\}$ and $\mathbb{M}(\boldsymbol{\theta}) = \mathbb{E}\{m(\boldsymbol{W}_t, \boldsymbol{\theta})\}$ for any $\boldsymbol{\theta} \in \Theta$.

APPENDIX A: AUXILIARY LEMMAS

In this section, we provide some useful lemmas that will be constantly used in the proofs of main results.

A.1. Lemmas for moment inequalities and empirical processes. The following lemma establishes a uniform law of large numbers for the segmented linear models with an α-mixing sequence of observations.

LEMMA A.1 (Glivenko-Cantelli). *Let $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1^{\mathrm{T}}, \boldsymbol{\gamma}_2^{\mathrm{T}})^{\mathrm{T}} \in \prod_{l=1}^2 \Gamma_l$. Let $U_t = U(\boldsymbol{W}_t)$ be a function of $\boldsymbol{W}_t$ with $\sup_t \mathbb{E}\|U_t\|^4 < \infty$. Then under the α-mixing condition in Assumption 1, for each $k \in \{1,\cdots,4\}$ we have*

$$\sup_{\boldsymbol{\gamma} \in \prod_{l=1}^2 \Gamma_l} |\mathbb{E}_T\{U_t \mathbb{1}\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma})\}\} - \mathbb{E}\{U_t \mathbb{1}\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma})\}\}| = o_p(1).$$

REMARK A.1. In this lemma, the geometric decaying rate of the α-mixing coefficient in Assumption 1 can be relaxed as a polynomial rate satisfying $\sum_{t=1}^{\infty} \alpha(t)^{1-\frac{2}{r}} < \infty$ for some $r > 2$.

PROOF. Let $\mathcal{F}_l = \{\boldsymbol{z}_l : \boldsymbol{z}_l^{\mathrm{T}}\boldsymbol{\gamma} < 0, \boldsymbol{\gamma} \in \Gamma_l\}$. By Example 2.6.1 of van der Vaart and Wellner (1996) we know that the VC-dimension of $\mathcal{F}_l$ is $\text{VC}(\mathcal{F}_l) = d_l$, where d_l is the dimension of $\boldsymbol{z}_l$ for $l = 1$ and 2. Let $R_k = \{R_k(\boldsymbol{\gamma}), \boldsymbol{\gamma} \in \prod_{l=1}^2 \Gamma_l\}$. Then, R_k consists of intersection of sets in $\{\mathcal{F}_l, l \in \{1,2\}\}$ or their complements. Then, according to Lemma 2.6.17 of van der Vaart and Wellner (1996), R_k is a VC-class which can pick out at most $O(n^{\sum_{l=1}^2 d_l - 2})$ subsets of any given set $\{\mathbf{x}_i\}_{i=1}^n$ for $\mathbf{x}_i \in \mathbb{R}^{\sum_{l=1}^2 d_l}$. Hence, by Lemma 2.6.18 of van der Vaart and Wellner (1996), the function class $\mathcal{G}_k = \{g(u,\boldsymbol{z}) = u\mathbb{1}(\boldsymbol{z} \in R), R \in R_k\}$ is a VC-subgraph function class, which implies that $\mathcal{G}_k$ has a finite uniform covering numbers.

For any fixed $\boldsymbol{\gamma} \in \prod_{l=1}^2 \Gamma_l$, by the ergodic thoeorem for the α-mixing processes (see Theorem 10.2.1 of Doob, 1953), we have $|\mathbb{E}_T\{U_t \mathbb{1}\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma})\}\} - \mathbb{E}\{U_t \mathbb{1}\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma})\}\}| = o_p(1)$ for each $k \in [4]$. Because the covering number of $\mathcal{G}_k$ is finite, using the same arguments as in Theorem 2.4.1 of van der Vaart and Wellner (1996), the uniform weak law of large numbers is established. □

The next lemma provides useful moment inequalities about perturbations of $\boldsymbol{\gamma}_0$ around its neighborhoods.

LEMMA A.2. *Suppose that U is a random variable that satisfies $M_0 < \mathbb{E}(U|\boldsymbol{Z}_\ell^{\mathrm{T}}\boldsymbol{\gamma} = 0) < M_1$ almost surely with some constants $M_0, M_1 > 0$ for any $\ell \in \{1,2\}$, where $\boldsymbol{\gamma} \in \mathcal{N}(\boldsymbol{\gamma}_{\ell 0}; \delta)$ for some $\delta > 0$.*

(i) Under Assumption 3.(ii), there exist constants $c_1, \delta_1 > 0$, such that if $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \mathcal{N}(\boldsymbol{\gamma}_{\ell 0}; \delta_1)$, then

$$\mathbb{E}\left\{U\left|\mathbb{1}_\ell(\boldsymbol{\gamma}_1) - \mathbb{1}_\ell(\boldsymbol{\gamma}_2)\right|\right\} \le c_1\|\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2\|. \tag{A.1}$$

(ii) Under Assumption 4.(i), there exist constants $c_2, \delta_2 > 0$, such that if $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \mathcal{N}(\boldsymbol{\gamma}_{\ell 0}; \delta_2)$, then

$$\mathbb{E}\left\{U\mathbb{1}(\boldsymbol{Z}_\ell \in R)\left|\mathbb{1}_\ell(\boldsymbol{\gamma}_{\ell 0}) - \mathbb{1}_\ell(\boldsymbol{\gamma}_\ell)\right|\right\} \ge c_2\|\boldsymbol{\gamma}_{\ell 0} - \boldsymbol{\gamma}_\ell\|. \tag{A.2}$$

where $R = R_k(\boldsymbol{\gamma}_0) \cup R_h(\boldsymbol{\gamma}_0)$ with $(k,h) \in \mathcal{S}(\ell)$.

(iii) Under Assumption 4.(iii), there exist constants $c_3, \delta_3 > 0$, such that if $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \mathcal{N}(\boldsymbol{\gamma}_{10}; \delta_3)$ and $\boldsymbol{\gamma}_3, \boldsymbol{\gamma}_4 \in \mathcal{N}(\boldsymbol{\gamma}_{20}; \delta_3)$, then

$$\mathbb{E}\left\{U\left|\mathbb{1}_1(\boldsymbol{\gamma}_1) - \mathbb{1}_1(\boldsymbol{\gamma}_2)\right|\left|\left|\mathbb{1}_2(\boldsymbol{\gamma}_3) - \mathbb{1}_2(\boldsymbol{\gamma}_4)\right|\right\} \le c_3\|\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2\|\|\boldsymbol{\gamma}_3 - \boldsymbol{\gamma}_4\|. \tag{A.3}$$

PROOF. (i) Let $\delta_1 = \min(\delta, \delta_0)$, where δ_0 is specified in Assumption 3 (ii) and δ is in the assumption of Lemma A.2 (i). Denote $\mathcal{N}_{1\ell} = \mathcal{N}(\boldsymbol{\gamma}_{\ell 0}; \delta_1)$. Since for any $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \mathcal{N}_{1\ell}$, the event $|\mathbb{1}_\ell(\boldsymbol{\gamma}_1) - \mathbb{1}_\ell(\boldsymbol{\gamma}_2)| > 0$ implies that there exists $\boldsymbol{\gamma}_3 = \lambda\boldsymbol{\gamma}_1 + (1-\lambda)\boldsymbol{\gamma}_2$ with $\lambda \in (0,1)$ such that $\boldsymbol{Z}_\ell^{\mathrm{T}}\boldsymbol{\gamma}_3 = 0$, we have

$$\begin{aligned}\mathbb{E}\left\{U\left|\mathbb{1}_\ell(\boldsymbol{\gamma}_1) - \mathbb{1}_\ell(\boldsymbol{\gamma}_2)\right|\right\} &\le \mathbb{E}_{\boldsymbol{Z}_\ell}\left\{\sup_{\boldsymbol{\gamma}_3 \in \mathcal{N}_{1\ell}} \mathbb{E}(U|\boldsymbol{Z}_\ell^{\mathrm{T}}\boldsymbol{\gamma}_3 = 0)\left|\mathbb{1}_\ell(\boldsymbol{\gamma}_1) - \mathbb{1}_\ell(\boldsymbol{\gamma}_2)\right|\right\} \\ &\le M_1\mathbb{E}\left(\left|\mathbb{1}_\ell(\boldsymbol{\gamma}_1) - \mathbb{1}_\ell(\boldsymbol{\gamma}_2)\right|\right) \le c_1 M_1\|\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2\|,\end{aligned}$$

where the last inequality is due to Assumption 3.(ii), which verifies (A.1).

(ii) For each $\ell = 1$ and 2, let $\mathcal{N}_{2\ell} = \mathcal{N}(\boldsymbol{\gamma}_{\ell 0}; \delta)$. Let M_R be a positive constant such that $P_R = \mathbb{P}(\boldsymbol{Z}_\ell \in \mathcal{A}_R) > 0$, where $\mathcal{A}_R = \{\|\boldsymbol{Z}_\ell\| \le M_R, \boldsymbol{Z}_\ell \in R\}$. Then, for any $\boldsymbol{\gamma}_\ell \in \mathcal{N}_{2\ell}$, we have

$$\begin{aligned}&\mathbb{E}\left\{U\mathbb{1}(\boldsymbol{Z}_\ell \in R)\left|\mathbb{1}_\ell(\boldsymbol{\gamma}_\ell) - \mathbb{1}_\ell(\boldsymbol{\gamma}_{\ell 0})\right|\right\} \\ =&\mathbb{E}_{\boldsymbol{Z}_\ell}[\mathbb{E}(U|\boldsymbol{Z}_\ell)\{\left|\mathbb{1}_\ell(\boldsymbol{\gamma}_\ell) - \mathbb{1}_\ell(\boldsymbol{\gamma}_{\ell 0})\right|\mathbb{1}(\boldsymbol{Z}_\ell \in R)\}] \\ \ge&\mathbb{E}_{\boldsymbol{Z}_\ell}[\inf_{\boldsymbol{\gamma}_3 \in \mathcal{N}_{2\ell}} \mathbb{E}(U|\boldsymbol{Z}_\ell^{\mathrm{T}}\boldsymbol{\gamma}_3 = 0)\{\left|\mathbb{1}_\ell(\boldsymbol{\gamma}_\ell) - \mathbb{1}_\ell(\boldsymbol{\gamma}_{\ell 0})\right|\mathbb{1}(\boldsymbol{Z}_\ell \in R)\}] \\ \ge&M_0\mathbb{E}\left\{\left|\mathbb{1}_\ell(\boldsymbol{\gamma}_\ell) - \mathbb{1}_\ell(\boldsymbol{\gamma}_{\ell 0})\right|\mathbb{1}(\boldsymbol{Z}_\ell \in R)\right\}, \\ \ge&M_0\mathbb{E}\left\{\left|\mathbb{1}_\ell(\boldsymbol{\gamma}_\ell) - \mathbb{1}_\ell(\boldsymbol{\gamma}_{\ell 0})\right|\mathbb{1}(\|\boldsymbol{Z}_\ell\| \le M_R, \boldsymbol{Z}_\ell \in R)\right\} \\ =&M_0\mathbb{E}\left\{\mathbb{1}(|q_\ell| < |\boldsymbol{Z}_\ell^{\mathrm{T}}\Delta\boldsymbol{\gamma}_\ell|)\mathbb{1}(\|\boldsymbol{Z}_\ell\| \le M_R, \boldsymbol{Z}_\ell \in R)\right\} \\ =&M_0P_R\mathbb{E}\left\{\mathbb{1}(|q_\ell| < |\boldsymbol{Z}_\ell^{\mathrm{T}}\Delta\boldsymbol{\gamma}_\ell|) \mid \boldsymbol{Z}_\ell \in \mathcal{A}_R\right\},\end{aligned} \tag{A.4}$$

where $\Delta\boldsymbol{\gamma}_\ell = \boldsymbol{\gamma}_\ell - \boldsymbol{\gamma}_{\ell 0}$. Take $\delta_3 = \min(\delta_2/M_R, \delta)$, where δ_2 is specified in Assumption 4.(i). Then, for any $\boldsymbol{\gamma}_\ell \in \mathcal{N}(\boldsymbol{\gamma}_{\ell 0}; \delta_2)$, we have $|\boldsymbol{Z}_\ell^{\mathrm{T}}\Delta\boldsymbol{\gamma}_\ell| \le \delta_2$. Since the first elements of $\boldsymbol{\gamma}_{\ell 0}$ and $\boldsymbol{\gamma}_\ell$ are 1, $\boldsymbol{Z}_\ell^{\mathrm{T}}\Delta\boldsymbol{\gamma}_\ell = \boldsymbol{Z}_{-1,\ell}^{\mathrm{T}}\Delta\boldsymbol{\gamma}_{-1,\ell}$. Hence, by Assumption 4.(i),

$$\begin{aligned}&\mathbb{E}\left\{\mathbb{1}(|q_\ell| < |\boldsymbol{Z}_{-1,\ell}^{\mathrm{T}}\Delta\boldsymbol{\gamma}_{-1,\ell}|)|\boldsymbol{Z}_\ell \in \mathcal{A}_R\right\} \\ \ge&c_2\mathbb{E}\left(|\boldsymbol{Z}_{-1,\ell}^{\mathrm{T}}\Delta\boldsymbol{\gamma}_{-1,\ell}| \mid \boldsymbol{Z}_\ell \in \mathcal{A}_R\right) \\ \ge&c_2\|\Delta\boldsymbol{\gamma}_{-1,\ell}\|\inf_{\|\boldsymbol{\gamma}_{-1}\|=1} \mathbb{E}\left(|\boldsymbol{Z}_{-1,\ell}^{\mathrm{T}}\boldsymbol{\gamma}_{-1}| \mid \boldsymbol{Z}_\ell \in \mathcal{A}_R\right)\end{aligned}$$

$$=c_2\|\boldsymbol{\gamma}_{\ell 0}-\boldsymbol{\gamma}_\ell\| \inf_{\|\boldsymbol{\gamma}_{-1}\|=1} \mathbb{E}\left(|\boldsymbol{Z}_{-1,\ell}^{\mathrm{T}}\boldsymbol{\gamma}_{-1}| \mid \boldsymbol{Z}_\ell \in \mathcal{A}_R\right). \tag{A.5}$$

We next show that $\inf_{\|\boldsymbol{\gamma}_{-1}\|=1} \mathbb{E}\left(|\boldsymbol{Z}_{-1,\ell}^{\mathrm{T}}\boldsymbol{\gamma}_{-1}| \mid \boldsymbol{Z}_\ell \in \mathcal{A}_R\right) > 0$. If otherwise, there exists some $\boldsymbol{\gamma}_*$ such that $\|\boldsymbol{\gamma}_{-1,*}\| = 1$ and $\mathbb{E}\left(|\boldsymbol{Z}_{-1,\ell}^{\mathrm{T}}\boldsymbol{\gamma}_{-1,*}|| \mid \boldsymbol{Z}_{-1,\ell} \in \mathcal{A}_R\right) = 0$. This means that $\mathbb{P}(|\boldsymbol{Z}_{-1,\ell}^{\mathrm{T}}\boldsymbol{\gamma}_{-1,*}| = 0|\boldsymbol{Z}_\ell \in \mathcal{A}_R) = 1$, which further implies that $\mathbb{P}(|\boldsymbol{Z}_{-1\ell}^{\mathrm{T}}\boldsymbol{\gamma}_{-1,*}| = 0) \geq \mathbb{P}(|\boldsymbol{Z}_{-1,\ell}^{\mathrm{T}}\boldsymbol{\gamma}_{-1,*}| = 0|\boldsymbol{Z}_\ell \in \mathcal{A}_R)\mathbb{P}(\boldsymbol{Z}_\ell \in \mathcal{A}_R) = P_R$, and contradicts with Assumption 3.(ii). Therefore, it must hold that

$$\inf_{\|\boldsymbol{\gamma}_{-1}\|=1} \mathbb{E}\left(|\boldsymbol{Z}_{-1,\ell}^{\mathrm{T}}\boldsymbol{\gamma}_{-1}| \mid \boldsymbol{Z}_\ell \in \mathcal{A}_R\right) > 0. \tag{A.6}$$

Combining (A.4)–(A.6) completes the proof of Part (ii) of Lemma A.2.

(iii) It follows from similar arguments as in (i) and thus is omitted. □

The following moment inequalities are for partial sums, built upon Lemma A.2 and Rosenthal-type moment inequalities for mixing sequences provided in Peligrad (1982).

LEMMA A.3 (Moment inequalities). *Let $U_t = U(\boldsymbol{W}_t)$ be a function of $\boldsymbol{W}_t$. Under Assumptions 1.(i), 3.(ii) and 4.(iii), and suppose that $\sup_{\boldsymbol{\gamma}\in\mathcal{N}(\boldsymbol{\gamma}_{l0};\delta_l)} \mathbb{E}(|U_t|^4 \mid \boldsymbol{Z}_{l,t}^{\mathrm{T}}\boldsymbol{\gamma} = 0) < M$ for almost surely $\boldsymbol{Z}_{l,t}$ for each $l = 1$ and 2, where δ_1 and M are positive constants. Then, there exist constants $c_1, c_2 > 0$ such that for each $l \in \{1, 2\}$, if $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \mathcal{N}(\boldsymbol{\gamma}_{l0}; \delta_l)$, then*

$$\mathbb{E}\left|\mathbb{G}_T\left[U_t\left\{\mathbb{1}_{l,t}(\boldsymbol{\gamma}_1) - \mathbb{1}_{l,t}(\boldsymbol{\gamma}_2)\right\}\right]\right|^4 \leq c_1\|\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2\|^2 \tag{A.7}$$

and if $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \mathcal{N}(\boldsymbol{\gamma}_{10}; \delta_1)$ and $\boldsymbol{\gamma}_3, \boldsymbol{\gamma}_4 \in \mathcal{N}(\boldsymbol{\gamma}_{20}; \delta_1)$, then

$$\mathbb{E}\left|\mathbb{G}_T\left[U_t\left\{\mathbb{1}_{1,t}(\boldsymbol{\gamma}_1) - \mathbb{1}_{1,t}(\boldsymbol{\gamma}_2)\right\}\left\{\mathbb{1}_{2,t}(\boldsymbol{\gamma}_3) - \mathbb{1}_{2,t}(\boldsymbol{\gamma}_4)\right\}\right]\right|^4 \leq c_2\|\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2\|^2\|\boldsymbol{\gamma}_3 - \boldsymbol{\gamma}_4\|^2. \tag{A.8}$$

PROOF. Denote by $U_t\{\mathbb{1}_{l,t}(\boldsymbol{\gamma}_1) - \mathbb{1}_{l,t}(\boldsymbol{\gamma}_2)\} = \widetilde{U}_t(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$. Then according to Lemma 3.6 of Peligrad (1982), there is a constant $C > 0$ such that

$$\begin{aligned}
&\mathbb{E}\left|\sum_{t=1}^{T}\{\widetilde{U}_t(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2) - \mathbb{E}\widetilde{U}_t(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)\}\right|^4 \\
\leq& C\left(T^2\|\widetilde{U}_t(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)\|_2^4 + T\|\widetilde{U}_t(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)\|_4^4\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathbb{E}\left|\mathbb{G}_T\{\widetilde{U}_t(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)\}\right|^4 \leq& 2C[\mathbb{E}\{\widetilde{U}_t(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)\}^2]^2 \\
=& 2C\{\mathbb{E}(U_t^2|\mathbb{1}_{l,t}(\boldsymbol{\gamma}_1) - \mathbb{1}_{l,t}(\boldsymbol{\gamma}_2)|)\}^2 \\
\leq& C'\|\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2\|^2,
\end{aligned} \tag{A.9}$$

for some constant $C' > 0$, where the last inequality is from (A.1) in Lemma A.2. Therefore, (A.7) is verified. Similarly, (A.8) can be shown by using Lemma 3.6 of Peligrad (1982) and the moment inequality (A.3). □

The next lemma is a maximal inequality for empirical processes with regime indicators under the α-mixing condition.

LEMMA A.4 (Maximal inequalities). *Suppose that the conditions in Lemma A.3 hold. Then there exist constants $c_1, c_2 > 0$ such that for any λ and $\varepsilon > 0$, it holds that*

$$\mathbb{P}\left\{\sup_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2\in\mathcal{N}(\boldsymbol{\gamma}_{l0};\varepsilon)} \left|\mathbb{G}_T\left[U_t\left\{\mathbb{1}_{l,t}(\boldsymbol{\gamma}_1)-\mathbb{1}_{l,t}(\boldsymbol{\gamma}_2)\right\}\right]\right| > \lambda\right\} \le \frac{c_1}{\lambda^2}\varepsilon^2, \text{ for } l=1,2 \text{ and} \tag{A.10}$$

$$\mathbb{P}\left\{\sup_{\substack{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2\in\mathcal{N}(\boldsymbol{\gamma}_{10};\varepsilon)\\ \boldsymbol{\gamma}_3,\boldsymbol{\gamma}_4\in\mathcal{N}(\boldsymbol{\gamma}_{20};\varepsilon)}} \left|\mathbb{G}_T\left[U_t\left\{\mathbb{1}_{1,t}(\boldsymbol{\gamma}_1)-\mathbb{1}_{1,t}(\boldsymbol{\gamma}_2)\right\}\left\{\mathbb{1}_{2,t}(\boldsymbol{\gamma}_3)-\mathbb{1}_{2,t}(\boldsymbol{\gamma}_4)\right\}\right]\right| > \lambda\right\} \le \frac{c_2}{\lambda^4}\varepsilon^4. \tag{A.11}$$

PROOF. The first part (A.10) follows similar arguments as that in proof of Lemma I.1 of Lee et al. (2021). We now show (A.11) by adapting the proof of Lemma I.1 of Lee et al. (2021), which mainly employed Theorem 1 of Bickel and Wichura (1971).

First, by applying (A.8) of Lemma A.3, we know that for some $\delta > 0$ and any $\boldsymbol{\gamma}_j, \boldsymbol{\gamma}_j' \in \mathcal{N}(\boldsymbol{\gamma}_{j0};\delta)$ and any $\boldsymbol{\gamma}_k, \boldsymbol{\gamma}_k' \in \mathcal{N}(\boldsymbol{\gamma}_{k0};\delta)$,

$$\mathbb{E}\left|\mathbb{G}_T\left\{U_t\left|\mathbb{1}_{j,t}(\boldsymbol{\gamma}_j)-\mathbb{1}_{j,t}(\boldsymbol{\gamma}_j')\right|\left|\mathbb{1}_{k,t}(\boldsymbol{\gamma}_k)-\mathbb{1}_{k,t}(\boldsymbol{\gamma}_k')\right|\right\}\right|^4 \le C_1\|\boldsymbol{\gamma}_j-\boldsymbol{\gamma}_j'\|^2\|\boldsymbol{\gamma}_k-\boldsymbol{\gamma}_k'\|^2, \tag{A.12}$$

for some constant $C_1 > 0$. Let $\boldsymbol{\gamma}_0 = (\boldsymbol{\gamma}_{j0}^{\mathrm{T}}, \boldsymbol{\gamma}_{k0}^{\mathrm{T}})^{\mathrm{T}}$, $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_j^{\mathrm{T}}, \boldsymbol{\gamma}_k^{\mathrm{T}})^{\mathrm{T}}$ and

$$J_T(\boldsymbol{\gamma}) = \mathbb{G}_T\left\{U_t\left|\mathbb{1}_{j,t}(\boldsymbol{\gamma}_j)-\mathbb{1}_{j,t}(\boldsymbol{\gamma}_{j0})\right|\left|\mathbb{1}_{k,t}(\boldsymbol{\gamma}_k)-\mathbb{1}_{k,t}(\boldsymbol{\gamma}_{k0})\right|\right\}. \tag{A.13}$$

By equation (1) of Bickel and Wichura (1971),

$$\sup_{\boldsymbol{\gamma}:\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_0\|\le\varepsilon} |J_T(\boldsymbol{\gamma})| \le d\cdot M'' + |J_T(\widetilde{\boldsymbol{\gamma}})|, \tag{A.14}$$

where $d = d_j + d_k$ and $\widetilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma}_0 + \varepsilon\mathbf{1}$ is the elementwise increment of $\boldsymbol{\gamma}_0$ by a positive constant ε, and the supremum is taken over a hyper-cube $\{\boldsymbol{\gamma} : 0 \le \gamma_i - \gamma_{i,0} \le \varepsilon, i \in [d]\}$, and the precise definition and an upper bound of M'' are referred to Bickel and Wichura (1971). It is sufficient to show that each of M'' and $J_T(\widetilde{\boldsymbol{\gamma}})$ satisfies the conclusion of the lemma since $|a|+|b| > 2c$ implies either $|a| > c$ or $|b| > c$.

To apply Theorem 1 of Bickel and Wichura (1971), we need to consider the increment of the process J_T around a block in the tube $T_\varepsilon = \{\boldsymbol{\gamma} : \|\boldsymbol{\gamma}-\boldsymbol{\gamma}_0\| \le \varepsilon\}$. For a block $B = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2] = (\gamma_{11}, \gamma_{21}] \times \cdots \times (\gamma_{1d}, \gamma_{2d}]$ in the tube T_ε , let

$$\begin{aligned} J_T(B) &= \sum_{k_1=0,1}\cdots\sum_{k_d=0,1}(-1)^{d-k_1-\cdots k_d} J_T\left(\gamma_{11}+k_1(\gamma_{21}-\gamma_{11}),\cdots,\gamma_{1d}+k_d(\gamma_{2d}-\gamma_{1d})\right) \\ &= \sum_{k_2=0,1}\cdots\sum_{k_d=0,1}(-1)^{d-k_2-\cdots k_d}\left\{J_T\left(\gamma_{11},\gamma_{12}+k_2(\gamma_{22}-\gamma_{12})\cdots,\gamma_{1d}+k_d(\gamma_{2d}-\gamma_{1d})\right)\right. \\ &\qquad \left. - J_T\left(\gamma_{21},\gamma_{12}+k_2(\gamma_{22}-\gamma_{12})\cdots,\gamma_{1d}+k_d(\gamma_{2d}-\gamma_{1d})\right)\right\}. \end{aligned}$$

It follows from the C_r-inequality that there exists some positive constants C_2 and C_3 such that

$$\begin{aligned} \mathbb{E}\left|J_T(B)\right|^4 \le & C_2\sum_{k_2=0,1}\cdots\sum_{k_d=0,1}\mathbb{E}\{\left|J_T\left(\gamma_{11},\gamma_{12}+k_2(\gamma_{22}-\gamma_{12})\cdots,\gamma_{1d}+k_d(\gamma_{2d}-\gamma_{1d})\right)\right. \\ & \left. - J_T\left(\gamma_{21},\gamma_{12}+k_2(\gamma_{22}-\gamma_{12})\cdots,\gamma_{1d}+k_d(\gamma_{2d}-\gamma_{1d})\right)\right|^4\}. \end{aligned} \tag{A.15}$$

Let $\boldsymbol{\gamma}_1(\boldsymbol{\psi}) = (\gamma_{11}, \boldsymbol{\psi}^{\mathrm{T}})^{\mathrm{T}}$ and $\boldsymbol{\gamma}_2(\boldsymbol{\psi}) = (\gamma_{21}, \boldsymbol{\psi}^{\mathrm{T}})^{\mathrm{T}}$, which are identical except for the first element, such that $\|\boldsymbol{\psi} - \boldsymbol{\gamma}_{-1,0}\| \leq \varepsilon$. Then, (A.15) implies that

$$\mathbb{E}\left|J_T(B)\right|^4 \leq C_3 \sup_{\boldsymbol{\psi} \in \mathcal{N}(\boldsymbol{\gamma}_{-1,0}; \varepsilon)} \mathbb{E}\left|J_T\left\{\boldsymbol{\gamma}_1(\boldsymbol{\psi})\right\} - J_T\left\{\boldsymbol{\gamma}_2(\boldsymbol{\psi})\right\}\right|^4 \tag{A.16}$$

for some positive constant C_3. Let $\widetilde{\boldsymbol{\psi}}_k$ be the last d_k elements $\boldsymbol{\psi}$, and let $\boldsymbol{\gamma}_{1,j}(\boldsymbol{\psi})$ and $\boldsymbol{\gamma}_{2,j}(\boldsymbol{\psi})$ be the vectors of the first d_j elements of $\boldsymbol{\gamma}_1(\boldsymbol{\psi})$ and $\boldsymbol{\gamma}_2(\boldsymbol{\psi})$, respectively. Then, note that for any $\boldsymbol{\psi}$, by the triangle inequality,

$$\begin{aligned}
&\left|J_T\left\{\boldsymbol{\gamma}_1(\boldsymbol{\psi})\right\} - J_T\left\{\boldsymbol{\gamma}_2(\boldsymbol{\psi})\right\}\right|^4 \\
&\leq \left|\mathbb{G}_T\left\{\left|U_t\right| \left|\mathbb{1}_{j,t}(\boldsymbol{\gamma}_{1,j}(\boldsymbol{\psi})) - \mathbb{1}_{j,t}(\boldsymbol{\gamma}_{2,j}(\boldsymbol{\psi}))\right| \left|\mathbb{1}_{k,t}(\widetilde{\boldsymbol{\psi}}_k) - \mathbb{1}_{k,t}(\boldsymbol{\gamma}_{k0})\right|\right\}\right|^4.
\end{aligned} \tag{A.17}$$

Since $\|\boldsymbol{\gamma}_{1,j}(\boldsymbol{\psi}) - \boldsymbol{\gamma}_{2,j}(\boldsymbol{\psi})\| \leq |\gamma_{11} - \gamma_{21}|$ and $\|\widetilde{\boldsymbol{\psi}}_k - \boldsymbol{\gamma}_{k0}\| \leq \varepsilon$ for any $\|\boldsymbol{\psi} - \boldsymbol{\gamma}_0\| \leq \varepsilon$, it follows from (A.12), (A.16), and (A.17) that there exists some poistive constant C_4 such that

$$\mathbb{E}\left|J_T(B)\right|^p \leq C_4 \left|\gamma_{11} - \gamma_{21}\right|^2 \varepsilon^2 \leq C_5 \left|\gamma_{11} - \gamma_{21}\right|^4,$$

where $C_5 \geq C_4 \varepsilon / \left|\gamma_{11} - \gamma_{21}\right|$. Now, without loss of generality, we can assume that $\mu(B) \geq C_5 \left|\gamma_{11} - \gamma_{21}\right|^d$, where μ denotes the Legesque measure in $\mathbb{R}^d$, since we can derive the same bound by choosing the smallest side length of B as $|\gamma_{11} - \gamma_{21}|$. This implies that $\mathbb{E}\left|J_T(B)\right|^4 \leq C_5 \left\{\mu(B)\right\}^{\frac{p}{d}}$ for any block $B \subset T_\varepsilon$. Therefore, we can take $\gamma_1 = \gamma_2 = 2$ and $\beta_1 = \beta_2 = \frac{2}{d}$ in the equation (3) of Bickel and Wichura (1971), implying that their equation (2) holds with $\gamma = 4$ and $\beta = \frac{4}{d}$. Since $\mu(T_\varepsilon) = \varepsilon^d$, by Theorem 1 of Bickel and Wichura (1971), we conclude that for any λ,

$$\mathbb{P}(M'' > \lambda) \leq \frac{C_6}{\lambda^4} \varepsilon^4, \tag{A.18}$$

for some positive constant C_6. Furthermore, by the Markov inequality and the moment bound in (A.12), there exists some positive constant C_7 such that

$$\mathbb{P}\left\{J_T(\widetilde{\boldsymbol{\gamma}}) > \lambda\right\} \leq \frac{C_7}{\lambda^4} \varepsilon^4. \tag{A.19}$$

Therefore, (A.11) is proved by combining (A.14), (A.18), and (A.19). This completes the proof of Lemma A.4. □

LEMMA A.5. *Suppose that the conditions in Lemma A.3 hold. Then we have*

$$\sup_{\|\boldsymbol{\gamma}_l - \boldsymbol{\gamma}_{l0}\| \lesssim T^{-1}} \sqrt{T} \mathbb{E}_T\left\{U_t \left|\mathbb{1}_{l,t}(\boldsymbol{\gamma}_l) - \mathbb{1}_{l,t}(\boldsymbol{\gamma}_{l0})\right|\right\} = o_p(1), \text{ for } l = 1, 2 \text{ and} \tag{A.20}$$

$$\sup_{\substack{\|\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_{10}\| \lesssim T^{-1} \\ \|\boldsymbol{\gamma}_2 - \boldsymbol{\gamma}_{20}\| \lesssim T^{-1}}} T \mathbb{E}_T\left\{U_t \left|\mathbb{1}_{1,t}(\boldsymbol{\gamma}_1) - \mathbb{1}_{1,t}(\boldsymbol{\gamma}_{10})\right| \left|\mathbb{1}_{2,t}(\boldsymbol{\gamma}_2) - \mathbb{1}_{2,t}(\boldsymbol{\gamma}_{20})\right|\right\} = o_p(1). \tag{A.21}$$

PROOF. For each $l = 1$ and 2, letting $\varepsilon = cT^{-1}$ in (A.10) for some constant $c > 0$ implies that

$$\sup_{\|\boldsymbol{\gamma}_l - \boldsymbol{\gamma}_{l0}\| \lesssim T^{-1}} \sqrt{T}(\mathbb{E}_T - \mathbb{E})\left\{U_t \left|\mathbb{1}_{l,t}(\boldsymbol{\gamma}_l) - \mathbb{1}_{l,t}(\boldsymbol{\gamma}_{l0})\right|\right\} = O_p\left(T^{-\frac{2}{p}}\right),$$

for $p \in (4, 4 + \beta)$ with β specified in Lemma A.3. According to (A.1) in Lemma A.2,

$$\sup_{\|\boldsymbol{\gamma}_l - \boldsymbol{\gamma}_{l0}\| \lesssim T^{-1}} \sqrt{T} \mathbb{E}\left\{U_t \left|\mathbb{1}_{l,t}(\boldsymbol{\gamma}_l) - \mathbb{1}_{l,t}(\boldsymbol{\gamma}_{l0})\right|\right\} = O\left(T^{-1}\right).$$

Combining the above two equalities leads to (A.20). Similarly, letting $\varepsilon = cT^{-1}$ in (A.11) for some constant $c > 0$ implies that

$$\sup_{\substack{\|\gamma_1-\gamma_{10}\|\lesssim T^{-1}\\ \|\gamma_2-\gamma_{20}\|\lesssim T^{-1}}} \sqrt{T}(\mathbb{E}_T - \mathbb{E})\left\{U_t\left|\mathbb{1}_{1,t}(\gamma_1) - \mathbb{1}_{1,t}(\gamma_{10})\right|\left|\mathbb{1}_{2,t}(\gamma_2) - \mathbb{1}_{2,t}(\gamma_{20})\right|\right\} = O_p\left(T^{-\frac{4}{p}}\right).$$

According to (A.2) in Lemma A.2 we have

$$\sup_{\substack{\|\gamma_1-\gamma_{10}\|\lesssim T^{-1}\\ \|\gamma_2-\gamma_{20}\|\lesssim T^{-1}}} \mathbb{E}\left\{U_t\left|\mathbb{1}_{1,t}(\gamma_1) - \mathbb{1}_{1,t}(\gamma_{10})\right|\left|\mathbb{1}_{2,t}(\gamma_2) - \mathbb{1}_{2,t}(\gamma_{20})\right|\right\} = O_p\left(T^{-2}\right).$$

Combining the above two equations leads to (A.21). □

LEMMA A.6. *Under the conditions of Lemma A.3, for any constatnts $\lambda, c_1, c_2 > 0$ and $j \neq k \in \{1, \cdots, 4\}$, we have*

$$\sup_{c_1T^{-1}\le\|\gamma-\gamma_0\|\le c_2}\left\{\left|(\mathbb{E}_T - \mathbb{E})\left(U_t\mathbb{1}_t^{(j)}(\gamma_0)\mathbb{1}_t^{(k)}(\gamma)\right)\right| - \lambda\|\gamma-\gamma_0\|\right\} = O_p(T^{-1}). \quad \text{(A.22)}$$

PROOF. The event that $j \neq k$ can be classed into two cases: (i) $(j,k) \in \mathcal{S}(i)$ for $i = 1$ or 2; and (ii) $(j,k) \notin \mathcal{S}(i)$ for both $i = 1$ and 2.

Case (i): $(j,k) \in \mathcal{S}(i)$ for $i \in \{1,2\}$. Without loss of generality, we take $j = 1, k = 2$ to illustrate. Note that

$$\begin{aligned}\mathbb{1}_t^{(1)}(\gamma_0)\mathbb{1}_t^{(2)}(\gamma) =& \mathbb{1}(\boldsymbol{Z}_{1,t}^{\mathrm{T}}\gamma_1 \le 0 < \boldsymbol{Z}_{1,t}^{\mathrm{T}}\gamma_{10})\mathbb{1}(\boldsymbol{Z}_{2,t}^{\mathrm{T}}\gamma_{20} > 0, \boldsymbol{Z}_{2,t}^{\mathrm{T}}\gamma_2 > 0)\\ =& \mathbb{1}(\boldsymbol{Z}_{1,t}^{\mathrm{T}}\gamma_1 \le 0 < \boldsymbol{Z}_{1,t}^{\mathrm{T}}\gamma_{10})\left\{\mathbb{1}(\boldsymbol{Z}_{2,t}^{\mathrm{T}}\gamma_{20} > 0) - \mathbb{1}(\boldsymbol{Z}_{2,t}^{\mathrm{T}}\gamma_2 \le 0 < \boldsymbol{Z}_{2,t}^{\mathrm{T}}\gamma_{20})\right\},\end{aligned}$$

which implies that

$$\begin{aligned}&\left|(\mathbb{E}_T - \mathbb{E})\left\{U_t\mathbb{1}_t^{(1)}(\gamma_0)\mathbb{1}_t^{(2)}(\gamma)\right\}\right| \le \left|(\mathbb{E}_T - \mathbb{E})\left\{\tilde{U}_t\mathbb{1}(\boldsymbol{Z}_{1,t}^{\mathrm{T}}\gamma_1 \le 0 < \boldsymbol{Z}_{1,t}^{\mathrm{T}}\gamma_{10})\right\}\right|\\ &+\left|(\mathbb{E}_T - \mathbb{E})\left\{U_t\mathbb{1}(\boldsymbol{Z}_{1,t}^{\mathrm{T}}\gamma_1 \le 0 < \boldsymbol{Z}_{1,t}^{\mathrm{T}}\gamma_{10})\mathbb{1}(\boldsymbol{Z}_{2,t}^{\mathrm{T}}\gamma_2 \le 0 < \boldsymbol{Z}_{2,t}^{\mathrm{T}}\gamma_{20})\right\}\right| =: I_{1,T}(\gamma) + I_{2,T}(\gamma), \text{ say,}\end{aligned}$$

where $\tilde{U}_t = U_t\mathbb{1}\left(\boldsymbol{Z}_{2,t}^{\mathrm{T}}\gamma_{20} > 0\right)$. Define the "shells"

$$S_{T,j} = \left\{\gamma : c_1jT^{-1} \le \|\gamma-\gamma_0\| < c_1(j+1)T^{-1}\right\}.$$

Then, for any $M > 0$, we have

$$\begin{aligned}&\mathbb{P}\left(\sup_{c_1T^{-1}\le\|\gamma-\gamma_0\|\le c_2} T\left\{I_{1,T}(\gamma) - \lambda\|\gamma-\gamma_0\|/2\right\} > M\right)\\ \le&\sum_{j=1}^{\infty}\mathbb{P}\left\{\gamma \in S_{T,j}, \left|(\mathbb{E}_T - \mathbb{E})\left\{\tilde{U}_t\mathbb{1}(\boldsymbol{Z}_{1,t}^{\mathrm{T}}\gamma_1 \le 0 < \boldsymbol{Z}_{1,t}^{\mathrm{T}}\gamma_{10})\right\}\right| > MT^{-1} + \lambda\|\gamma-\gamma_0\|/2\right\}\\ \le&\sum_{j=1}^{\infty}\mathbb{P}\left\{\gamma \in S_{T,j}, \left|(\mathbb{E}_T - \mathbb{E})\left\{\tilde{U}_t\mathbb{1}(\boldsymbol{Z}_{1,t}^{\mathrm{T}}\gamma_1 \le 0 < \boldsymbol{Z}_{1,t}^{\mathrm{T}}\gamma_{10})\right\}\right| > (M + c_1j\lambda/2)T^{-1}\right\}\\ \le&\sum_{j=1}^{\infty}\frac{c_3(j+1)^2}{(M + c_1j\lambda/2)^4} = O\left(\frac{1}{M^4}\right), \quad \text{(A.23)}\end{aligned}$$

where the last inequality is by invoking (A.10) in Lemma A.4. Via the similar argument, we obtain

$$\mathbb{P}\left(\sup_{c_1T^{-1}\le\|\gamma-\gamma_0\|\le c_2} T\left\{I_{2,T}(\gamma) - \lambda\|\gamma-\gamma_0\|/2\right\} > M\right) = O\left(\frac{1}{T^2M^4}\right). \quad \text{(A.24)}$$

This together with (A.23) verifies (A.22).

Case (ii): $(j,k) \notin \mathcal{S}(i)$ for either $i = 1$ or 2. Without loss of generality, we take $j = 1, k = 3$ to illustrate. Then

$$\mathbb{1}_t^{(1)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(3)}(\boldsymbol{\gamma}) = \mathbb{1}(\boldsymbol{Z}_{1,t}^{\mathrm{T}}\boldsymbol{\gamma}_1 \le 0 < \boldsymbol{Z}_{1,t}^{\mathrm{T}}\boldsymbol{\gamma}_{10})\mathbb{1}(\boldsymbol{Z}_{2,t}^{\mathrm{T}}\boldsymbol{\gamma}_2 \le 0 < \boldsymbol{Z}_{2,t}^{\mathrm{T}}\boldsymbol{\gamma}_{20}). \tag{A.25}$$

Therefore, $\left|(\mathbb{E}_T - \mathbb{E})\left\{U_t\mathbb{1}_t^{(1)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(3)}(\boldsymbol{\gamma})\right\}\right| = I_2(\boldsymbol{\gamma})$ and the result follows from (A.24). Combining the two cases completes the proof for the lemma. □

A.2. Lemmas for Poisson point processes. We first introduce some basic notations for the point measures and point processes following the definitions in Resnick (2008).

DEFINITION A.2 (Point measures). Suppose that E is a locally compact space with a countable basis whose Borel σ-algebra of subsets is $\mathcal{E}$. A *point process* on E is a measure m of the following form: for $\{\mathbf{x}_i, i \ge 1\}$, which is a countable collection of points of E, and any Borel set $A \in \mathcal{E}$, $m(A) := \sum_{i=1} \mathbb{1}(\mathbf{x}_i \in A)$. If $m(K) < \infty$ for any compact set $K \in \mathcal{E}$, then m is said to be Radon. Let $M_p(E)$ be the space of all Radon point measures on E. A sequence $\{m_n\} \subset M_p(E)$ is said to converge vaguely to m, if $\int_E f dm_n \to \int_E f dm$ as $n \to \infty$ for all $f \in C_K(E)$, the continuous function space with compact support K. The vague convergence induces a vague topology on $M_p(E)$. Topological space $M_p(E)$ is then metrizable as a complete separable metric space. Define $\mathcal{M}_p(E)$ as the σ-algebra generated by open sets in $M_p(E)$.

DEFINITION A.3 (Point processes and their weak convergence). A *point process* on E is a measurable map from a probability space $(\Omega, \mathcal{A}, \mathbb{P}) \to (M_p(E), \mathcal{M}_p(E))$, i.e., for every event $\omega \in \Omega$, the realization of the point process $\mathbf{N}(\omega)$ is some point measure in $M_p(E)$. A sequence of point processes $\mathbf{N}_n$ *weak converges* of a point process $\mathbf{N}$, denoted as $\mathbf{N}_n \Rightarrow \mathbf{N}$ if $\mathbb{E}_{\mathbb{P}}\{h(\mathbf{N}_n)\} \to \mathbb{E}_{\mathbb{P}}\{h(\mathbf{N})\}$ for all continuous and bounded functions h mapping $M_p(E)$ to $\mathbb{R}$. Note that if $\mathbf{N}_n \Rightarrow \mathbf{N}$ then $\int_E f(\mathbf{x})d\mathbf{N}_n(\boldsymbol{X}) \xrightarrow{d} \int_E f(\mathbf{x})d\mathbf{N}(\boldsymbol{X})$ for any $f \in C_K(E)$ by the continuous mapping theorem.

DEFINITION A.4 (Poisson point process). A point process $\mathbf{N}$ is called a *Poisson process measure* (PRM) with mean measure μ if $\mathbf{N}$ satisfies

(i) for any $F \in \mathcal{E}$ and any non-negative integer k, $\mathbb{P}(\mathbf{N}(F) = k) = \exp\{-\mu(F)\}\{\mu(F)\}^k/k!$ if $\mu(F) < \infty$ and $\mathbb{P}(\mathbf{N}(F) = k) = 0$ if $\mu(F) = \infty$;

(ii) if $F_1, \cdots, F_k$ are mutually disjoint sets in $\mathcal{E}$, then $\{\mathbf{N}(F_i), i \le k\}$ are independent random variables.

The following two lemmas, from Proposition 3.22 of Resnick (2008) and Theorem 1 of Meyer (1973), respectively, provide key tools to study the weak convergence of point processes of extreme events with α-mixing time series.

LEMMA A.7 (Kallenberg's theorem). *Suppose that* $\mathbf{N}$ *is a point process on* E *and* $\mathcal{T}$ *is a basis of relatively compact open sets such that* $\mathcal{T}$ *is closed under finite unions and intersections, and for any* $F \in \mathcal{T}$, $\mathbb{P}\{\mathbf{N}(\partial F) = 0\} = 1$. *Then* $\widehat{\mathbf{N}}_T \Rightarrow \mathbf{N}$ *if for all* $F \in \mathcal{T}$,

$$\lim_{T\to\infty} \mathbb{P}\left\{\widehat{\mathbf{N}}_T(F) = 0\right\} = \mathbb{P}\{\mathbf{N}(F) = 0\}, \quad \textit{and} \tag{A.26}$$

$$\lim_{T\to\infty} \mathbb{E}\left\{\widehat{\mathbf{N}}_T(F)\right\} = \mathbb{E}\{\mathbf{N}(F)\} < \infty. \tag{A.27}$$

LEMMA A.8 (Meyer's theorem). *Suppose that the sequence* $\{A_t^n\}_{t=1}^n\,(n=1,2,\cdots)$ *is stationary and* α*-mixing with mixing coefficient* $\alpha_n(k)$ *defined as*

$$\alpha_n(k)=\sup_{\substack{E\in\Omega_1^m,\\ F\in\Omega_{m+k+1}^n}}\left|\mathbb{P}(EF)-\mathbb{P}(E)\mathbb{P}(F)\right|,\ \textit{where}\ \Omega_j^J=\sigma(A_j^n,\cdots,A_J^n)\ 1\le j<J\le n$$

for any $1\le k\le n$. *Suppose that the probability of the event* A_t^n *is* $\mathbb{P}(A_t^n)=\frac{a}{n}+o(\frac{1}{n})$ *for some* $a>0$. *Moreover, suppose that the following conditions hold: there exist sequences of block sizes* $\{p_m,m\ge1\}$, $\{q_m,m\ge1\}$ *and* $\{t_m=m(p_m+q_m),m\ge1\}$ *such that*
(a) for any $r>0$, $m^r\alpha_{t_m}(q_m)\to0$ *as* $m\to\infty$, *where* $t_m=m(p_m+q_m)$,
(b) $q_m/p_m\to0, p_{m+1}/p_m\to1$ *as* $m\to\infty$, *and*
(c) $I_{p_m}=\sum_{i=1}^{p_m-i}(p_m-i)\mathbb{P}(A_1^{t_m}\cap A_{i+1}^{t_m})=o(\frac{1}{m})$ *as* $m\to\infty$.
Then it holds that

$$\mathbb{P}\left(\textit{exactly } k \textit{ events among } \{A_t^n\}_{t=1}^n \textit{ happen}\right)\to\frac{e^{-a}a^k}{k!}\quad \textit{as } n\to\infty.$$

Remark. (i) Note that for any given $n<\infty$, the α-mixing coefficient $\alpha_n(k)$ defined above is upper bounded by the commonly used α-mixing coefficient $\alpha(k)$ (see e.g., Doukhan, 1995), where the supreme of F is taken over Ω_{m+k+1}^∞ instead of Ω_{m+k+1}^n. (ii) The proof of the above theorem is based on partitioning the observations into consecutive blocks of size p_m and q_m alternately. The condition $I_{p_m}=o(1/m)$ prevents clusters of rare events A_t^n, preventing the compound Poisson processes as the limit.

A.3. Lemmas for epi-convergence. In the investigation of the limiting distribution of $\widehat{\gamma}$ and $\widehat{\boldsymbol{\beta}}$, we will employ the tool of epi-convergence in distribution (Knight, 1999), which is useful in establishing weak convergences of "argmin" functionals, and is more general than uniform convergence, because it allows for more general discontinuity.

DEFINITION A.5 (Epi-convergence in distribution). Suppose that $\{Q_n(\mathbf{x})\}$ is a sequence of random lower semi-continuous (l-sc) functions, namely $Q_n(\mathbf{x})\le\liminf_{\mathbf{x}_j\to x}Q_n(\mathbf{x}_j)$ for any $\mathbf{x}$ and any sequence $\{\mathbf{x}_j\}$ whose limit is $\mathbf{x}$. Let $\mathcal{L}$ be the space of l-sc functions $f:\mathbb{R}^d\to\bar{\mathbb{R}}$, where $\bar{\mathbb{R}}=[-\infty,\infty]$. The space $\mathcal{L}$ can be made into a complete separable metric space. (Rockafellar and Wets, 1998).

A sequence of functions $\{Q_n\}\in\mathcal{L}$ is said to epi-converge in distribution to Q if for any closed rectangles $R_1,\cdots,R_k$ in $\mathbb{R}^d$ with open interiors $R_1^\circ,\cdots,R_k^\circ$, and any real numbers $r_1,\cdots,r_k$:

$$\begin{aligned}\mathbb{P}(\cap_{j=1}^k\{\inf_{\mathbf{x}\in R_j}Q(\mathbf{x})>r_j\})&\le\liminf_{n\to\infty}\mathbb{P}(\cap_{j=1}^k\{\inf_{\mathbf{x}\in R_j}Q_n(\mathbf{x})>r_j\})\\&\le\limsup_{n\to\infty}\mathbb{P}(\cap_{j=1}^k\{\inf_{\mathbf{x}\in R_j^\circ}Q_n(\mathbf{x})>r_j\})\\&\le\mathbb{P}(\cap_{j=1}^k\{\inf_{\mathbf{x}\in R_j^\circ}Q(\mathbf{x})>r_j\}).\end{aligned}$$

The above definition of the epi-convergence can be difficult to verify. Instead, we will use an equivalent characterization given by Knight (1999), using the finite-dimensional convergence and stochastic equi-lower-semicontinuouity.

DEFINITION A.6 (Finite-dimensional convergence in distribution). A sequence of random functions $\{Q_n(\mathbf{x})\}$ converges to $Q(\mathbf{x})$ in distribution in the finite-dimensional sense if for any finite positive integer k and any $(\mathbf{x}_1, \cdots, \mathbf{x}_k)$, it holds that

$$(Q_n(\mathbf{x}_1), \cdots, Q_n(\mathbf{x}_k)) \xrightarrow{d} (Q(\mathbf{x}_1), \cdots, Q(\mathbf{x}_k)).$$

DEFINITION A.7 (Stochastic equi-lower-semicontinuous). A sequence $\{Q_n\} \in \mathcal{L}$, where $\mathcal{L}$ is the space of l-sc functions defined in Definition A.5, is said to be stochastic equi-lower-semicontinuous (s.e-l-sc), if for any compact set B and any $\epsilon, \delta > 0$, there exists $\mathbf{x}_1, \cdots, \mathbf{x}_k \in B$, for a finite integer k, and some open sets $\{V(\mathbf{x}_i)\}_{i=1}^k$ covering B and containing $\mathbf{x}_1, \cdots, \mathbf{x}_k$, such that

$$\limsup_{n\to\infty} \mathbb{P}\left(\cup_{j=1}^k \left\{ \inf_{\mathbf{x}\in V(\mathbf{x}_j)} Q_n(\mathbf{x}) \le \min(\epsilon^{-1}, Q_n(\mathbf{x}_j) - \epsilon) \right\} \right) < \delta.$$

LEMMA A.9 (Theorem 2 of Knight, 1999). *Let $\{Q_n\}$ be a stochastic e-l-sc sequence of functions. Then $\{Q_n\}$ converges to Q in distribution in the finite-dimensional sense if and only if $\{Q_n\}$ epi-converges in distribution to Q.*

APPENDIX B: PROOFS FOR SECTION 3

B.1. Proof of Proposition 1. The following proof is for Proposition 1 on the identification of $\boldsymbol{\theta}_0$.

PROOF. Note that $\mathbb{M}(\boldsymbol{\theta})$ can be expanded as

$$\begin{aligned}
&\mathbb{M}(\boldsymbol{\theta}) = \mathbb{E}\{m(\boldsymbol{W}, \boldsymbol{\theta})\} \\
=&\mathbb{E}(\varepsilon^2) + \mathbb{E}[\sum_{k=1}^4 \sum_{h=1}^4 \{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_h - \boldsymbol{\beta}_{k0})\}^2 \mathbb{1}^{(k)}(\boldsymbol{\gamma}_0)\mathbb{1}^{(h)}(\boldsymbol{\gamma})] \\
&+ 2\mathbb{E}\{\sum_{k=1}^4 \sum_{h=1}^4 \varepsilon \boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_h - \boldsymbol{\beta}_{k0})\mathbb{1}^{(k)}(\boldsymbol{\gamma}_0)\mathbb{1}^{(h)}(\boldsymbol{\gamma})\} \\
=&\mathbb{E}(\varepsilon^2) + \sum_{k=1}^4 \sum_{h=1}^4 \mathbb{E}[\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_h - \boldsymbol{\beta}_{k0})\}^2 \mathbb{1}^{(k)}(\boldsymbol{\gamma}_0)\mathbb{1}^{(h)}(\boldsymbol{\gamma})] \\
=&\mathbb{M}(\boldsymbol{\theta}_0) + \sum_{k=1}^4 \sum_{h=1}^4 A_{k,h}(\boldsymbol{\theta}), \quad \text{say,}
\end{aligned} \tag{B.1}$$

where the second equality is because of $\mathbb{E}(\varepsilon|\boldsymbol{X}, \boldsymbol{Z}) = 0$. If $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$, then one of the following two cases will hold: (1): $\boldsymbol{\gamma} \neq \boldsymbol{\gamma}_0$, or (2): $\boldsymbol{\gamma} = \boldsymbol{\gamma}_0$ while $\boldsymbol{\beta} \neq \boldsymbol{\beta}_0$. We now consider the two cases respectively.

Case (1). Suppose that $\boldsymbol{\gamma} \neq \boldsymbol{\gamma}_0$. Then for some $l \in \{1, 2\}$ and $h \in \{1, \cdots, 4\}$, the true splitting hyperplane $H_{l0} : \boldsymbol{z}_l^{\mathrm{T}}\boldsymbol{\gamma}_{l0} = 0$ will partition through $R_h(\boldsymbol{\gamma})$. Because Assumption 2 (i) implies that $\mathbb{P}\{|q_l| < \epsilon|\boldsymbol{Z}_{-1,l}\} > 0$ almost surely for any $\epsilon > 0$, meaning there is a positive probability that $\boldsymbol{Z}$ will locate around the neighborhood of the hyperplane $\boldsymbol{z}_l^{\mathrm{T}}\boldsymbol{\gamma}_{l0} = 0$, we have that for some $(k, j) \in \mathcal{S}(l)$, it holds that $\mathbb{P}\{\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0) \cap R_h(\boldsymbol{\gamma})\} > 0$ and $\mathbb{P}\{\boldsymbol{Z} \in R_j(\boldsymbol{\gamma}_0) \cap R_h(\boldsymbol{\gamma})\} > 0$. Therefore,

$$A_{k,h}(\boldsymbol{\theta}) \ge \lambda_0 \|\boldsymbol{\beta}_h - \boldsymbol{\beta}_{k0}\|^2, \;\; A_{j,h}(\boldsymbol{\theta}) \ge \lambda_0 \|\boldsymbol{\beta}_h - \boldsymbol{\beta}_{j0}\|^2$$

according to Assumption 2 (ii). Since $\boldsymbol{\beta}_{k0} \neq \boldsymbol{\beta}_{j0}$, either $A_{k,h}(\boldsymbol{\theta}) > 0$ or $A_{j,h}(\boldsymbol{\theta}) > 0$. Consequently, $\mathbb{M}(\boldsymbol{\theta}) \geq \mathbb{M}(\boldsymbol{\theta}_0) + A_{k,h}(\boldsymbol{\theta}) + A_{j,h}(\boldsymbol{\theta}) > \mathbb{M}(\boldsymbol{\theta}_0)$.

Case (2). Suppose that $\boldsymbol{\gamma} = \boldsymbol{\gamma}_0$ while $\boldsymbol{\beta}_{k0} \neq \boldsymbol{\beta}_k$ for some $k \in \{1, \cdots, 4\}$. In such a case,

$$A_{k,k}(\boldsymbol{\theta}) = \mathbb{E}\left[\{\boldsymbol{X}_t^{\mathrm{T}}(\boldsymbol{\beta}_k - \boldsymbol{\beta}_{k0})\}^2 \mathbb{1}\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma}_0)\}\right] \geq \lambda_0 \|\boldsymbol{\beta}_k - \boldsymbol{\beta}_{k0}\|^2 > 0,$$

by Assumption 2 (ii). Therefore, $\mathbb{M}(\boldsymbol{\theta}) \geq \mathbb{M}(\boldsymbol{\theta}_0) + A_{k,k}(\boldsymbol{\theta}) > \mathbb{M}(\boldsymbol{\theta}_0)$. Combining the two cases yields that $\mathbb{M}(\boldsymbol{\theta}) > \mathbb{M}(\boldsymbol{\theta}_0)$ if $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$, which completes the proof. □

B.2. Proof of Theorem 3.1. The following proof is for Theorem 3.1 on the consistency of $\widehat{\boldsymbol{\theta}}$.

PROOF. The consistency of $\widehat{\boldsymbol{\theta}}$ follows the standard approach for M-estimation (van der Vaart, 1998). First, we strengthen the result of Proposition 3.1 by a separable condition (B.2), which can be induced by the continuity of $\mathbb{M}(\boldsymbol{\theta})$ at $\boldsymbol{\theta}_0$. Note that $\mathbb{M}(\boldsymbol{\theta}) = \mathbb{E}(Y^2) - 2\sum_{k=1}^4 \mathbb{E}\{Y\boldsymbol{X}^{\mathrm{T}}\boldsymbol{\beta}_k \mathbb{1}(\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}))\} + \sum_{k=1}^4 \mathbb{E}\{(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{\beta}_k)^2 \mathbb{1}(\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}))\}$. The continuity with respect to $\boldsymbol{\beta}$ is obvious and it remains to show the continuity at $\boldsymbol{\gamma}_0$. Note that for any $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$,

$$\begin{aligned}
&\left|\mathbb{E}\{(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{\beta})^2 \mathbb{1}(\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}))\} - \mathbb{E}\{(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{\beta})^2 \mathbb{1}(\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0))\}\right| \\
\leq &\mathbb{E}^{1/2}\{(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{\beta})^4\} |\mathbb{E}\{\mathbb{1}(\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}))\} - \mathbb{E}\{\mathbb{1}(\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0))\}|^{1/2} \\
\leq &\mathbb{E}^{1/2}\{(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{\beta})^4\} \{\sum_{l=1}^2 |\mathbb{P}(\boldsymbol{Z}_l^{\mathrm{T}}\boldsymbol{\gamma}_l < 0) - \mathbb{P}(\boldsymbol{Z}_l^{\mathrm{T}}\boldsymbol{\gamma}_{l0} < 0)|\}^{1/2} \lesssim \sqrt{\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_0\|},
\end{aligned}$$

where the last inequality is due to Assumption 3.(ii). Thus, $\mathbb{M}(\boldsymbol{\theta})$ is continuous at $\boldsymbol{\theta}_0$, implying that

$$\sup_{\|\boldsymbol{\theta} - \boldsymbol{\theta}_0\| > \epsilon} \mathbb{M}(\boldsymbol{\theta}) > \mathbb{M}(\boldsymbol{\theta}_0) \ \ \forall \epsilon > 0. \tag{B.2}$$

As a direct consequence of Lemma A.1. we have the following uniform convergence

$$\sup_{\boldsymbol{\theta} \in \Theta} |\mathbb{M}(\boldsymbol{\theta}) - \mathbb{M}_T(\boldsymbol{\theta})| \xrightarrow{P} 0, \tag{B.3}$$

as $T \to \infty$. By the definition of $\widehat{\boldsymbol{\theta}}$, we have $\mathbb{M}_T(\widehat{\boldsymbol{\theta}}) \leq \mathbb{M}_T(\boldsymbol{\theta}_0) + o_p(1)$. Because (B.3) implies that $\mathbb{M}_T(\boldsymbol{\theta}_0) \xrightarrow{P} \mathbb{M}(\boldsymbol{\theta}_0)$. It follows that $\mathbb{M}_T(\widehat{\boldsymbol{\theta}}) \leq \mathbb{M}(\boldsymbol{\theta}_0) + o_p(1)$, whence

$$\begin{aligned}
\mathbb{M}(\widehat{\boldsymbol{\theta}}) - \mathbb{M}(\boldsymbol{\theta}_0) \leq &\mathbb{M}(\widehat{\boldsymbol{\theta}}) - \mathbb{M}_T(\widehat{\boldsymbol{\theta}}) + o_p(1) \\
\leq &\sup_{\boldsymbol{\theta} \in \Theta} |\mathbb{M}(\boldsymbol{\theta}) - \mathbb{M}_T(\boldsymbol{\theta})| + o_p(1) \xrightarrow{P} 0.
\end{aligned} \tag{B.4}$$

Because of (B.2), for any $\epsilon > 0$, there exists $\eta > 0$ such that $\mathbb{M}(\boldsymbol{\theta}) > \mathbb{M}(\boldsymbol{\theta}_0) + \eta$ if $\|\boldsymbol{\theta} - \boldsymbol{\theta}_0\| > \epsilon$. Thus, the event $\left\{\|\widehat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}_0\| > \epsilon\right\}$ is contained in the event $\left\{\mathbb{M}(\widehat{\boldsymbol{\theta}}) > \mathbb{M}(\boldsymbol{\theta}_0) + \eta\right\}$, whose probability converges to 0 in view of (B.4), which completes the proof for $\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\| \xrightarrow{P} 0$ as $T \to \infty$. □

B.3. Proof of Corollary 3.1.

PROOF. Let $\mathcal{D}_T = \{\boldsymbol{W}_t\}_{t=1}^T$. We prove the corollary for $k = 1$ without loss of generality, where $R_1(\boldsymbol{\gamma}_0) = \{\boldsymbol{z}_l^{\mathrm{T}}\boldsymbol{\gamma}_{l0} > 0, l = 1 \text{ and } 2\}$. Then $R_1(\boldsymbol{\gamma}_0) \setminus R_1(\boldsymbol{\gamma})$ is a subset of $\cup_{l=1}^2 \{\boldsymbol{z} : \boldsymbol{z}_l^{\mathrm{T}}\boldsymbol{\gamma}_{l0} > 0 > \boldsymbol{z}_l^{\mathrm{T}}\boldsymbol{\gamma}_l\}$. Therefore,

$$\begin{aligned}\mathbb{P}\{\boldsymbol{Z} \in R_1(\boldsymbol{\gamma}_0) \setminus R_1(\widehat{\boldsymbol{\gamma}})|\mathcal{D}_T\} &\le \sum_{l=1}^2 \mathbb{P}(\boldsymbol{Z}_l^{\mathrm{T}}\boldsymbol{\gamma}_{l0} > 0 > \boldsymbol{Z}_l^{\mathrm{T}}\widehat{\boldsymbol{\gamma}}_l|\mathcal{D}_T) \\ &\le c_1 \sum_{l=1}^2 \|\widehat{\boldsymbol{\gamma}}_l - \boldsymbol{\gamma}_{l0}\|, \end{aligned} \tag{B.5}$$

where the probability is taken over $\boldsymbol{Z}$, and the second inequality is due to the consistency of $\widehat{\boldsymbol{\gamma}}$ and Assumption 3.(ii). Therefore, $\mathbb{P}\{\boldsymbol{Z} \in R_1(\boldsymbol{\gamma}_0) \setminus R_1(\widehat{\boldsymbol{\gamma}})|\mathcal{D}_T\} \to 0$ as $T \to \infty$. Similarly, we have $\mathbb{P}\{\boldsymbol{Z} \in R_1(\widehat{\boldsymbol{\gamma}}) \setminus R_1(\boldsymbol{\gamma}_0)|\mathcal{D}_T\} \to 0$. Since $\mathbb{P}\{\boldsymbol{Z} \in R_1(\boldsymbol{\gamma}_0) \triangle R_1(\widehat{\boldsymbol{\gamma}})|\mathcal{D}_T\} = \mathbb{P}\{\boldsymbol{Z} \in R_1(\boldsymbol{\gamma}_0) \setminus R_1(\widehat{\boldsymbol{\gamma}})|\mathcal{D}_T\} + \mathbb{P}\{\boldsymbol{Z} \in R_1(\widehat{\boldsymbol{\gamma}}) \setminus R_1(\boldsymbol{\gamma}_0)|\mathcal{D}_T\}$, we obtain

$$\mathbb{P}\{\boldsymbol{Z} \in R_1(\boldsymbol{\gamma}_0) \triangle R_1(\widehat{\boldsymbol{\gamma}})|\mathcal{D}_T\} \xrightarrow{P} 0$$

as $T \to \infty$. Because $\mathbb{P}\{\boldsymbol{Z} \in R_1(\boldsymbol{\gamma}_0) \triangle R_1(\widehat{\boldsymbol{\gamma}})|\mathcal{D}_T\}$ is uniformly integrable, we have

$$\mathbb{P}\{\boldsymbol{Z} \in R_1(\boldsymbol{\gamma}_0) \triangle R_1(\widehat{\boldsymbol{\gamma}})\} = \mathbb{E}_{\mathcal{D}_T}[\mathbb{P}\{\boldsymbol{Z} \in R_1(\boldsymbol{\gamma}_0) \triangle R_1(\widehat{\boldsymbol{\gamma}})|\mathcal{D}_T\}] \to 0,$$

which completes the proof. □

B.4. Proof of Theorem 3.2. The following proof is for Theorem 3.2 on the convergence rate of $\widehat{\boldsymbol{\theta}}$.

PROOF. The convergence rate will be derived in two steps. In the first step, we establish that there is a metric d such that

$$d^2(\boldsymbol{\theta}, \boldsymbol{\theta}_0) \lesssim \mathbb{E}\{m(\boldsymbol{W}_t, \boldsymbol{\theta}) - m(\boldsymbol{W}_t, \boldsymbol{\theta}_0)\} \text{ for any } \boldsymbol{\theta} \in \mathcal{N}(\boldsymbol{\theta}_0; \delta_0), \tag{B.6}$$

for some $\delta_0 > 0$. In the second step, we derive a convergence rate of $\mathbb{E}\{m(\boldsymbol{W}_t, \widehat{\boldsymbol{\theta}}) - m(\boldsymbol{W}_t, \boldsymbol{\theta}_0)\}$ by bounding $(\mathbb{E}_T - \mathbb{E})\{m(\boldsymbol{W}_t, \widehat{\boldsymbol{\theta}}) - m(\boldsymbol{W}_t, \boldsymbol{\theta}_0)\}$, which combined with Step 1 will lead to the desired convergence rate of $\widehat{\boldsymbol{\theta}}$.

Step 1. Note that we can decompose $\mathbb{E}\{m(\boldsymbol{W}_t, \boldsymbol{\theta}) - m(\boldsymbol{W}_t, \boldsymbol{\theta}_0)\}$ as

$$\begin{aligned}&\mathbb{E}\{m(\boldsymbol{W}_t, \boldsymbol{\theta}) - m(\boldsymbol{W}_t, \boldsymbol{\theta}_0)\} \\ =&\sum_{j=1}^4 \mathbb{E}\left\{(\boldsymbol{X}_t^{\mathrm{T}}(\boldsymbol{\beta}_{j0} - \boldsymbol{\beta}_j))^2 \mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}(\boldsymbol{\gamma})\right\} \\ &+ \sum_{i=1}^4 \sum_{k\neq i}^4 \mathbb{E}\left\{(\boldsymbol{X}_t^{\mathrm{T}}(\boldsymbol{\beta}_{i0} - \boldsymbol{\beta}_k))^2 \mathbb{1}_t^{(i)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(k)}(\boldsymbol{\gamma})\right\}, \\ =:&\sum_{j=1}^4 J_j(\boldsymbol{\theta}) + \sum_{i=1}^4 \sum_{k\neq i}^4 G_{ik}(\boldsymbol{\theta}), \quad \text{say,}\end{aligned} \tag{B.7}$$

where the $J_j(\boldsymbol{\theta})$ term corresponds to the part of observations which are classified to the jth region under both the hyperplanes with coefficient $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}$, and the G_{ik} term corresponds of the part of observations which are classified to the ith region under the hyperplanes with coefficient $\boldsymbol{\gamma}_0$, but classified to the kth region under $\boldsymbol{\gamma}$.

First, for each $j \in \{1, \cdots, 4\}$, note that

$$\begin{aligned}\mathbb{P}\{\boldsymbol{Z}_t \in R_j(\boldsymbol{\gamma}_0) \cap R_j(\boldsymbol{\gamma})\} =& \mathbb{P}\{\boldsymbol{Z}_t \in R_j(\boldsymbol{\gamma}_0)\} - \mathbb{P}\{\boldsymbol{Z}_t \in R_j(\boldsymbol{\gamma}_0) \setminus R_j(\boldsymbol{\gamma})\} \\ \overset{(i)}{\geq}& \mathbb{P}\{\boldsymbol{Z}_t \in R_j(\boldsymbol{\gamma}_0)\} - c_0\|\boldsymbol{\gamma}_0 - \boldsymbol{\gamma}\| \geq \mathbb{P}\{\boldsymbol{Z}_t \in R_j(\boldsymbol{\gamma}_0)\} - c_0\delta \\ \overset{(ii)}{\geq}& \mathbb{P}(\boldsymbol{Z}_t \in R_j(\boldsymbol{\gamma}_0))/2 > 0, \end{aligned} \tag{B.8}$$

uniformly for any $\boldsymbol{\gamma} \in \mathcal{N}(\boldsymbol{\gamma}_0; \delta)$, where (i) is due to (B.5) and (ii) is by taking δ sufficiently small, which is legitimate because of the consistency of $\widehat{\boldsymbol{\gamma}}$. Then by Assumption 2.(ii),

$$J_j(\boldsymbol{\theta}) \geq c_1\|\boldsymbol{\beta}_{j0} - \boldsymbol{\beta}_j\|^2. \tag{B.9}$$

For each $l \in \{1, 2\}$, we choose one pair $(i_l, k_l) \in \mathcal{S}(l)$. Without loss of generality, let $i_1 = 1, k_1 = 2, i_2 = 1, k_2 = 3$. We now bound the term $G_{i_l k_l}(\boldsymbol{\theta})$ from below,

$$\begin{aligned} G_{i_l k_l}(\boldsymbol{\theta}) =& \mathbb{E}\left\{\left(\boldsymbol{X}_t^{\mathrm{T}}\left(\boldsymbol{\beta}_{i_l 0} - \boldsymbol{\beta}_{k_l}\right)\right)^2 \mathbb{1}_t^{(i_l)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(k_l)}(\boldsymbol{\gamma})\right\} \\ =& \mathbb{E}\left\{\left(\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{i_l k_l, 0}\right)^2 \mathbb{1}_t^{(i_l)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(k_l)}(\boldsymbol{\gamma})\right\} + \mathbb{E}\left\{\left(\boldsymbol{X}_t^{\mathrm{T}}\left(\boldsymbol{\beta}_{k_l 0} - \boldsymbol{\beta}_{k_l}\right)\right)^2 \mathbb{1}_t^{(i_l)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(k_l)}(\boldsymbol{\gamma})\right\} \\ &+ 2\mathbb{E}\left\{\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{i_l k_l, 0}\boldsymbol{X}_t^{\mathrm{T}}\left(\boldsymbol{\beta}_{k_l 0} - \boldsymbol{\beta}_{k_l}\right)\mathbb{1}_t^{(i_l)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(k_l)}(\boldsymbol{\gamma})\right\} \\ \geq& \mathbb{E}\left\{\left(\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{i_l k_l, 0}\right)^2 \mathbb{1}_t^{(i_l)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(k_l)}(\boldsymbol{\gamma})\right\} \\ &- 2\mathbb{E}\left\{\left|\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{i_l k_l, 0}\right|\left|\boldsymbol{X}_t^{\mathrm{T}}\left(\boldsymbol{\beta}_{k_l 0} - \boldsymbol{\beta}_{k_l}\right)\right|\mathbb{1}_t^{(i_l)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(k_l)}(\boldsymbol{\gamma})\right\}. \end{aligned}$$

Similarly,

$$\begin{aligned} G_{k_l i_l}(\boldsymbol{\theta}) \geq& \mathbb{E}\left\{\left(\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{i_l k_l, 0}\right)^2 \mathbb{1}_t^{(k_l)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(i_l)}(\boldsymbol{\gamma})\right\} \\ &- 2\mathbb{E}\left\{\left|\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{i_l k_l, 0}\right|\left|\boldsymbol{X}_t^{\mathrm{T}}\left(\boldsymbol{\beta}_{i_l 0} - \boldsymbol{\beta}_{i_l}\right)\right|\mathbb{1}_t^{(k_l)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(i_l)}(\boldsymbol{\gamma})\right\}. \end{aligned}$$

Let $g_t^{i_l k_l} = \left(\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{i_l k_l, 0}\right)^2 \mathbb{1}\left\{\boldsymbol{Z}_t \in R_{i_l}(\boldsymbol{\gamma}_0) \cup R_{k_l}(\boldsymbol{\gamma}_0)\right\}$. Then

$$\left(\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{i_l k_l, 0}\right)^2 \mathbb{1}_t^{(i_l)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(k_l)}(\boldsymbol{\gamma}) + \left(\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{i_l k_l, 0}\right)^2 \mathbb{1}_t^{(k_l)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(i_l)}(\boldsymbol{\gamma}) = g_t^{i_l k_l}\left|\mathbb{1}_{l,t}(\boldsymbol{\gamma}_{l0}) - \mathbb{1}_{l,t}(\boldsymbol{\gamma}_l)\right|,$$

whose expectation is bounded by

$$\mathbb{E}\left\{g_t^{i_l k_l}\left|\mathbb{1}_{l,t}(\boldsymbol{\gamma}_{l0}) - \mathbb{1}_{l,t}(\boldsymbol{\gamma}_l)\right|\right\} \geq c_3\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\|, \tag{B.10}$$

for some constants $c_3 > 0$ due to Assumption 4 (ii) and Lemma A.2 (ii).

For the second term of the lower bound of $G_{i_l k_l}(\boldsymbol{\theta})$, note that there exists a positive constant c_4 such that

$$\begin{aligned} &\mathbb{E}\left\{\left|\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{i_l k_l, 0}\right|\left|\boldsymbol{X}_t^{\mathrm{T}}\left(\boldsymbol{\beta}_{k_l 0} - \boldsymbol{\beta}_{k_l}\right)\right|\mathbb{1}_t^{(i_l)}(\boldsymbol{\gamma}_0)\mathbb{1}_{l,t}(\boldsymbol{\gamma}_l, \boldsymbol{\gamma}_{l0})\right\} \\ &\leq \|\boldsymbol{\beta}_{k_l 0} - \boldsymbol{\beta}_{k_l}\|\|\boldsymbol{\delta}_{i_l k_l, 0}\|\mathbb{E}\left(\|\boldsymbol{X}_t\|^2 \mathbb{1}_{l,t}(\boldsymbol{\gamma}_l, \boldsymbol{\gamma}_{l0})\right) \\ &\leq c_4\|\boldsymbol{\beta}_{k_l 0} - \boldsymbol{\beta}_{k_l}\|\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\|, \end{aligned} \tag{B.11}$$

where the first inequality follows from the Cauchy-Schwartz inequality and the second is implied by Lemma A.2. Similarly,

$$\begin{aligned} &\mathbb{E}\left\{\left|\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{i_l k_l, 0}\right|\left|\boldsymbol{X}_t^{\mathrm{T}}\left(\boldsymbol{\beta}_{i_l 0} - \boldsymbol{\beta}_{i_l}\right)\right|\mathbb{1}_t^{(k_l)}(\boldsymbol{\gamma}_0)\mathbb{1}_{l,t}(\boldsymbol{\gamma}_l, \boldsymbol{\gamma}_{l0})\right\} \\ &\leq c_4\|\boldsymbol{\beta}_{i_l 0} - \boldsymbol{\beta}_{i_l}\|\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\|. \end{aligned} \tag{B.12}$$

Combining (B.10)–(B.12) leads to an lower bound of $G_{i_l k_l}(\boldsymbol{\theta}) + G_{k_l i_l}(\boldsymbol{\theta})$. For each given $l = 1$ and 2, these together with (B.9) with $j = k_l$ and i_l lead to

$$\begin{aligned}
&\{J_{k_l}(\boldsymbol{\theta}) + J_{i_l}(\boldsymbol{\theta})\}/2 + G_{i_l k_l}(\boldsymbol{\theta}) + G_{k_l i_l}(\boldsymbol{\theta}) \\
\geq & c_1(\|\boldsymbol{\beta}_{k_l 0} - \boldsymbol{\beta}_{k_l}\|^2 + \|\boldsymbol{\beta}_{i_l 0} - \boldsymbol{\beta}_{i_l}\|^2)/2 + c_3\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\| \\
& - 2c_4\left(\|\boldsymbol{\beta}_{k_l 0} - \boldsymbol{\beta}_{k_l}\| + \|\boldsymbol{\beta}_{i_l 0} - \boldsymbol{\beta}_{i_l}\|\right)\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\| \\
= & \sum_{j_l \in \{i_l, k_l\}} \left(\frac{c_1}{2}\|\boldsymbol{\beta}_{j_l 0} - \boldsymbol{\beta}_{j_l}\|^2 + \frac{c_3}{2}\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\| - 2c_4\|\boldsymbol{\beta}_{j_l 0} - \boldsymbol{\beta}_{j_l}\|\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\|\right) \\
= & \sum_{j_l \in \{i_l, k_l\}} L_j^l, \quad \text{say.}
\end{aligned} \tag{B.13}$$

A lower bound for the term L_j^l can be derived by considering the following two cases.
(i) If $c_1\|\boldsymbol{\beta}_{j_l 0} - \boldsymbol{\beta}_{j_l}\| \geq 8c_4\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\|$, then

$$L_j^l \geq \frac{c_1}{4}\|\boldsymbol{\beta}_{j_l 0} - \boldsymbol{\beta}_{j_l}\|^2 + \frac{c_3}{2}\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\|.$$

(ii) If $c_1\|\boldsymbol{\beta}_{j_l 0} - \boldsymbol{\beta}_{j_l}\| < 8c_4\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\|$, then

$$\frac{c_3}{2}\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\| - 2c_4\|\boldsymbol{\beta}_{j_l 0} - \boldsymbol{\beta}_{j_l}\|\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\| \geq \frac{c_3}{2}\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\| - 16\frac{c_4^2}{c_1} \cdot \|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\|^2,$$

which can be further bounded from below by $c_3\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\|/4$ provided that $\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\| \leq c_1 c_3/(64c_4^2)$, which is ensured by the consistency of $\widehat{\boldsymbol{\gamma}}$. Therefore, in the case (ii),

$$L_j^l \geq \frac{c_1}{2}\|\boldsymbol{\beta}_{j_l 0} - \boldsymbol{\beta}_{j_l}\|^2 + \frac{c_3}{4}\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\|,$$

provided that $\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\| \leq c_1 c_3/(64c_4^2)$. Combining Cases (i) and (ii), we have

$$L_j^l \geq c_5(\|\boldsymbol{\beta}_{j_l 0} - \boldsymbol{\beta}_{j_l}\|^2 + \frac{1}{2}\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\|),$$

for some generic constant $c_5 > 0$, as long as $\|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\| \leq c_4^2/(32c_1)$. By (B.13) we have

$$\begin{aligned}
&\{J_{k_l}(\boldsymbol{\theta}) + J_{i_l}(\boldsymbol{\theta})\}/2 + G_{i_l k_l}(\boldsymbol{\theta}) + G_{k_l i_l}(\boldsymbol{\theta}) \\
\geq & c_5(\|\boldsymbol{\beta}_{i_l 0} - \boldsymbol{\beta}_{i_l}\|^2 + \|\boldsymbol{\beta}_{k_l 0} - \boldsymbol{\beta}_{k_l}\|^2 + \|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\|),
\end{aligned} \tag{B.14}$$

for some positive constant c_5. Divide the regime index set $\{1, \cdots, 4\}$ to two parts: $\mathcal{K}_1 = \{k_l, i_l : l \in \{1, 2\}\}$ and $\mathcal{K}_2 = \{1, \cdots, 4\}/\mathcal{K}_1$. Then from (B.7), (B.9) and (B.14),

$$\begin{aligned}
\mathbb{M}(\boldsymbol{\theta}) - \mathbb{M}(\boldsymbol{\theta}_0) \geq & \sum_{k \in \mathcal{K}_1} J_j(\boldsymbol{\theta}) + \sum_{k \in \mathcal{K}_2} J_j(\boldsymbol{\theta}) + \sum_{i=1}^{K}\sum_{k \neq i}^{K} G_{ik}(\boldsymbol{\theta}) \\
\geq & \sum_{l=1}^{2}\{G_{i_l k_l}(\boldsymbol{\theta}) + G_{k_l i_l}(\boldsymbol{\theta}) + \frac{J_{k_l}(\boldsymbol{\theta}) + J_{i_l}(\boldsymbol{\theta})}{2}\} + \sum_{k \in \mathcal{K}_2} J_j(\boldsymbol{\theta}) \\
\geq & c_5 \sum_{l=1}^{2}(\|\boldsymbol{\beta}_{i_l 0} - \boldsymbol{\beta}_{i_l}\|^2 + \|\boldsymbol{\beta}_{k_l 0} - \boldsymbol{\beta}_{k_l}\|^2 + \|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\|) + c_1 \sum_{k \in \mathcal{K}_2} \|\boldsymbol{\beta}_{k0} - \boldsymbol{\beta}_k\|^2 \\
\geq & c_6(\sum_{k=1}^{4} \|\boldsymbol{\beta}_{k0} - \boldsymbol{\beta}_k\|^2 + \sum_{l=1}^{2} \|\boldsymbol{\gamma}_{l0} - \boldsymbol{\gamma}_l\|),
\end{aligned}$$

where $c_6 = \min\{c_1, c_5\}$. Finally, by the triangle inequality,

$$\mathbb{M}(\boldsymbol{\theta}) - \mathbb{M}(\boldsymbol{\theta}_0) \geq c_6(\|\boldsymbol{\beta}_0 - \boldsymbol{\beta}\|^2 + \|\boldsymbol{\gamma}_0 - \boldsymbol{\gamma}\|), \tag{B.15}$$

provided that $\boldsymbol{\gamma} \in \mathcal{N}(\boldsymbol{\gamma}_0; \delta_0)$ for some $\delta_0 > 0$. Denoting by $d(\boldsymbol{\theta}, \boldsymbol{\theta}_0) = \sqrt{\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_0\|} + \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|$ leads to the desired (B.6).

Step 2. Note that for any $\boldsymbol{\theta} \in \Theta$, we have

$$\begin{aligned}
&(\mathbb{E} - \mathbb{E}_T)\{m(\boldsymbol{W}_t, \boldsymbol{\theta})\} - (\mathbb{E} - \mathbb{E}_T)\{m(\boldsymbol{W}_t, \boldsymbol{\theta}_0)\} \\
=&\sum_{j=1}^{4}(\mathbb{E} - \mathbb{E}_T)[\{(\boldsymbol{X}_t^{\mathrm{T}}(\boldsymbol{\beta}_{j0} - \boldsymbol{\beta}_j))^2 \mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}(\boldsymbol{\gamma})\}] \\
&+ \sum_{i=1}^{4}\sum_{k\neq i}^{4}(\mathbb{E} - \mathbb{E}_T)[\{(\boldsymbol{X}_t^{\mathrm{T}}(\boldsymbol{\beta}_{i0} - \boldsymbol{\beta}_k))^2 \mathbb{1}_t^{(i)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(k)}(\boldsymbol{\gamma})\}] \\
&+ 2\sum_{j=1}^{4}\mathbb{E}_T[\{\varepsilon_t(\boldsymbol{X}_t^{\mathrm{T}}(\boldsymbol{\beta}_{j0} - \boldsymbol{\beta}_j))\mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}(\boldsymbol{\gamma})\}] \\
&+ 2\sum_{i=1}^{4}\sum_{k\neq i}^{4}\mathbb{E}_T[\{\varepsilon_t \boldsymbol{X}_t^{\mathrm{T}}(\boldsymbol{\beta}_{i0} - \boldsymbol{\beta}_k)\mathbb{1}_t^{(i)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(k)}(\boldsymbol{\gamma})\}] \\
=&S_{1,T} + S_{2,T} + S_{3,T} + S_{4,T}, \text{ say.}
\end{aligned} \tag{B.16}$$

We now bound the four terms respectively. For $S_{1,T}$, note that $\mathbb{1}_t^{(j)}(\boldsymbol{\gamma}) = 1 - \sum_{k\neq j}^{4}\mathbb{1}_t^{(k)}(\boldsymbol{\gamma})$ and

$$\begin{aligned}
S_{1,T} \leq& \sum_{j=1}^{4}\left|(\mathbb{E} - \mathbb{E}_T)\left\{(\boldsymbol{X}_t^{\mathrm{T}}(\boldsymbol{\beta}_{j0} - \boldsymbol{\beta}_j))^2 \mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}(\boldsymbol{\gamma})\right\}\right| \\
\leq& \sum_{j=1}^{4}\left|(\mathbb{E} - \mathbb{E}_T)\left\{(\boldsymbol{X}_t^{\mathrm{T}}(\boldsymbol{\beta}_{j0} - \boldsymbol{\beta}_j))^2 \mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)\right\}\right| \\
&+ \sum_{j=1}^{4}\sum_{k\neq j}^{4}(\mathbb{E} - \mathbb{E}_T)\left|\left\{(\boldsymbol{X}_t^{\mathrm{T}}(\boldsymbol{\beta}_{j0} - \boldsymbol{\beta}_j))^2 \mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(k)}(\boldsymbol{\gamma})\right\}\right| \\
=& S_{1,a,T} + S_{1,b,T}, \text{ say.}
\end{aligned}$$

For $S_{1,a,T}$, by the Cauchy-Schwartz inequality and the ULLN in Lemma A.1, we have $S_{1,a,T} = \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|^2 o_p(1)$. For $S_{1,b,T}$, due to the compactness of the parameter space for $\boldsymbol{\beta}_j$, Assumption 4 (iv) and Lemma A.6, it can be shown that $S_{1,b,T} = \lambda\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_0\| + O_p(T^{-1})$ for any $\lambda > 0$ and $\boldsymbol{\gamma} \in (c_1 T^{-1}, c_2)$ for any $c_1, c_2 > 0$. Therefore,

$$S_{1,T} \leq \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|^2 o_p(1) + \lambda\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_0\| + O_p(T^{-1}). \tag{B.17}$$

For the second term, we have

$$\begin{aligned}
S_{2,T} \leq& 2\sum_{i=1}^{4}\sum_{k\neq i}^{4}\left|(\mathbb{E} - \mathbb{E}_T)\left\{(\boldsymbol{X}_t^{\mathrm{T}}(\boldsymbol{\beta}_{i0} - \boldsymbol{\beta}_k))^2 \mathbb{1}_t^{(i)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(k)}(\boldsymbol{\gamma})\right\}\right| \\
=& \lambda\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_0\| + O_p(T^{-1}),
\end{aligned} \tag{B.18}$$

for any $\lambda > 0$, $\boldsymbol{\gamma} \in (c_1 T^{-1}, c_2)$, and any $c_1, c_2 > 0$, implied by the same reasoning for the $S_{1,b,T}$ term. For $S_{3,T}$, similar to $S_{1,T}$, it can be decomposed by

$$\begin{aligned} S_{3,T} \leq & 2\sum_{j=1}^{4} \left| \mathbb{E}_T \left\{ \varepsilon_t \left(\boldsymbol{X}_t^{\mathrm{T}} (\boldsymbol{\beta}_{j0} - \boldsymbol{\beta}_j) \right) \mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0) \right\} \right| \\ & + 2\sum_{j=1}^{4} \sum_{k \neq j}^{4} \left| \mathbb{E}_T \left\{ \varepsilon_t \left(\boldsymbol{X}_t^{\mathrm{T}} (\boldsymbol{\beta}_{j0} - \boldsymbol{\beta}_j) \right) \mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0) \mathbb{1}_t^{(k)}(\boldsymbol{\gamma}) \right\} \right| \\ = & S_{3,a,T} + S_{3,b,T}, \text{ say.} \end{aligned}$$

For $S_{3,a,T}$, by the martingale central limit theorem (Hall and Heyde, 1980) we have $S_{3,a,T} = \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\| O_p(T^{-1/2})$. For $S_{3,b,T}$, using the same arguments as that for $S_{1,b,T}$, $S_{3,b,T} = \lambda \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_0\| + O_p(T^{-1})$. Therefore,

$$S_{3,T} \leq \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\| O_p(T^{-1/2}) + \lambda \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_0\| + O_p(T^{-1}). \tag{B.19}$$

For $S_{4,T}$, following the same reasons for $S_{2,T}$, it can be shown that

$$S_{4,T} \leq \lambda \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_0\| + O_p(T^{-1}). \tag{B.20}$$

Putting (B.17)–(B.20) together, we obtain that if $\boldsymbol{\gamma} \in (c_1 T^{-1}, c_2)$ for some $c_1, c_2 > 0$, then

$$\begin{aligned} (\mathbb{E} - \mathbb{E}_T)\{m(\boldsymbol{W}_t, \boldsymbol{\theta}) - m(\boldsymbol{W}_t, \boldsymbol{\theta}_0)\} \leq & \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\| O_p(T^{-1/2}) + \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|^2 o_p(1) \\ & + 4\lambda \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_0\| + O_p(T^{-1}). \end{aligned}$$

Since $\mathbb{E}_T\{m(\boldsymbol{W}_t, \widehat{\boldsymbol{\theta}})\} \leq \mathbb{E}_T\{m(\boldsymbol{W}_t, \boldsymbol{\theta}_0)\}$ and (B.15), we obtain

$$\begin{aligned} C_6(\|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\|^2 + \|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|) \leq & \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\| O_p(T^{-1/2}) + \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|^2 o_p(1) \\ & + 4\lambda \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_0\| + O_p(T^{-1}). \end{aligned}$$

Since the above bound holds for any $\lambda \in (0, 1)$, we can take $\lambda < C_6/4$, which delivers

$$C_6 \|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\|^2 + (C_6 - 4\lambda)\|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\| \leq \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\| O_p(T^{-1/2}) + \|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\|^2 o_p(1) + O_p(T^{-1}),$$

which further implies $\|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\|^2 = O_p(T^{-1})$, and thus, $\|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\| = O_p(T^{-1})$. □

Proof of Corollary 3.2

PROOF. It can be seen straightforwardly from the proof of Corollary 3.1 that for each $k \in \{1, \cdots, 4\}$,

$$\mathbb{P}\{\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0) \triangle R_k(\widehat{\boldsymbol{\gamma}}) | \mathcal{D}_T\} \lesssim \sum_{l=1}^{2} \|\widehat{\boldsymbol{\gamma}}_l - \boldsymbol{\gamma}_{l0}\|, \tag{B.21}$$

which is of order $O_p(T^{-1/2})$ by Theorem 3.2. With the uniformly integratability of $\mathbb{P}\{\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0) \triangle R_k(\widehat{\boldsymbol{\gamma}}) \mid \mathcal{D}_T\}$, the conclusion of the corollary follows. □

B.5. Proof of Theorem 3.3. The following proof is for Theorem 3.3 on the asymptotic distribution of $\widehat{\boldsymbol{\theta}}$, which requires the following lemmas. Considering that the proofs for these lemmas are quite lengthy, we provide their proofs later in Subsections B.6–B.10.

For any $(\boldsymbol{u}^{\mathrm{T}}, \boldsymbol{v}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{4p+d_1+d_2}$, we define

$$Q_T(\boldsymbol{u}, \boldsymbol{v}) = \sum_{t=1}^{T} \{m(\boldsymbol{W}_t, \boldsymbol{\beta}_0 + \frac{\boldsymbol{u}}{\sqrt{T}}, \boldsymbol{\gamma}_0 + \frac{\boldsymbol{v}}{T}) - m(\boldsymbol{W}_t, \boldsymbol{\beta}_0, \boldsymbol{\gamma}_0)\}. \tag{B.22}$$

The following lemma establishes the separability for $Q_T(\boldsymbol{u}, \boldsymbol{v})$, whose proof is available in Section B.6.

LEMMA B.1. *Under Assumptions 1-5, uniformly for* $(\boldsymbol{u}^{\mathrm{T}}, \boldsymbol{v}^{\mathrm{T}})^{\mathrm{T}}$ *in any compact region of* $\mathbb{R}^{4p+d_1+d_2}$*, we have*

$$Q_T(\boldsymbol{u}, \boldsymbol{v}) = W_T(\boldsymbol{u}) + D_T(\boldsymbol{v}) + o_p(1), \tag{B.23}$$

where

$$W_T(\boldsymbol{u}) = \sum_{j=1}^{4} [\boldsymbol{u}_j^{\mathrm{T}} \mathbb{E}\{\boldsymbol{X}_t \boldsymbol{X}_t^{\mathrm{T}} \mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)\} \boldsymbol{u}_j - 2 \frac{\boldsymbol{u}_j^{\mathrm{T}}}{\sqrt{T}} \sum_{t=1}^{T} \boldsymbol{X}_t \varepsilon_t \mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)], \tag{B.24}$$

and

$$D_T(\boldsymbol{v}) = \sum_{t=1}^{T} \sum_{l=1}^{2} \sum_{(j,k) \in \mathcal{S}(l)} \xi_t^{(j,k)} \mathbb{1} \left\{ s_l^{(j)} \left(T q_{l,t} + \boldsymbol{Z}_{-1,l,t}^{\mathrm{T}} \boldsymbol{v}_{-1,l} \right) \leq 0 < s_l^{(j)} T q_{l,t} \right\}, \tag{B.25}$$

with

$$\xi_t^{(j,k)} = \left(\boldsymbol{\delta}_{jk,0}^{\mathrm{T}} \boldsymbol{X}_t \boldsymbol{X}_t^{\mathrm{T}} \boldsymbol{\delta}_{jk,0} + 2 \boldsymbol{X}_t^{\mathrm{T}} \boldsymbol{\delta}_{jk,0} \varepsilon_t \right) \{ \mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0) + \mathbb{1}_t^{(k)}(\boldsymbol{\gamma}_0) \},$$

where $\boldsymbol{\delta}_{jk,0} = \boldsymbol{\beta}_{j0} - \boldsymbol{\beta}_{k0}$*,* $q_{l,t} = \boldsymbol{Z}_{l,t}^{\mathrm{T}} \boldsymbol{\gamma}_{l0}$*,* $\mathcal{S}(l)$ *is the set of indices of adjacent regions split by the* l*-th hyperplane as defined in* (3)*, and* $s_l^{(j)} = \mathrm{sign}(\boldsymbol{z}_l^{\mathrm{T}} \boldsymbol{\gamma}_{l0})$ *for* $\boldsymbol{z} \in R_j(\boldsymbol{\gamma}_0)$ *as defined in* (2) *of the main text.*

The next lemma is to obtain the finite-dimensional weak limit of $D_T(\boldsymbol{v})$, whose behaviour is determined by the point processes induced by the observations which are near the splitting hyperplanes. The following notations are needed for this lemma and its proof. For each $l = 1, 2$ and $(j, k) \in \mathcal{S}(l)$, suppose $(q_l, \boldsymbol{Z}_{-1,l}, \xi^{(j,k)})$ follows the stationary distribution of $(q_{l,t}, \boldsymbol{Z}_{-1,l,t}, \xi_t^{(j,k)})$. We denote $F_{q_l|\boldsymbol{Z}_{-1,l}}(q|\boldsymbol{Z}_{-1,l})$ and $F_{\xi^{(j,k)}|q_l,\boldsymbol{Z}_{-1,l}}(\xi|q_l, \boldsymbol{Z}_{-1,l})$ as the conditional distributions of q_l on $\boldsymbol{Z}_{-1,l}$ and $\xi^{(j,k)}$ on $(q_l, \boldsymbol{Z}_{-1,l})$, respectively, and the corresponding conditional densities are $f_{q_l|\boldsymbol{Z}_{-1,l}}(q|\boldsymbol{Z}_{-1,l})$ and $f_{\xi^{(j,k)}|q_l,\boldsymbol{Z}_{-1,l}}(\xi|q_l, \boldsymbol{Z}_{-1,l})$, respectively. Let $\mathcal{Z}_{-1,l}$ be the compact support of the density of $\boldsymbol{Z}_{-1,l}$ as required in Assumption 5.

LEMMA B.2. *Under Assumptions 1-5, the finite-dimensional weak limit of* $D_T(\boldsymbol{v})$ *in* (B.25) *is*

$$D(\boldsymbol{v}) = \sum_{l=1}^{2} \sum_{j,k \in \mathcal{S}(l)} \sum_{i=1}^{\infty} \xi_i^{(j,k)} \mathbb{1} \left\{ s_l^{(j)} \left(J_{i,l}^{(j,k)} + (\boldsymbol{Z}_{l,i}^{(j,k)})^{\mathrm{T}} \boldsymbol{v}_{-1,l} \right) \leq 0 < s_l^{(j)} J_{i,l}^{(j,k)} \right\}, \tag{B.26}$$

for $\boldsymbol{v} = (\boldsymbol{v}_1^{\mathrm{T}}, \boldsymbol{v}_2^{\mathrm{T}})^{\mathrm{T}}$*, where* $\{(\xi_i^{(j,k)}, \boldsymbol{Z}_{l,i}^{(j,k)})\}_{i=1}^{\infty}$ *are independent copies of* $(\bar{\xi}^{(j,k)}, \boldsymbol{Z}_{-1,l})$ *with* $\bar{\xi}^{(j,k)} \sim F_{\xi^{(j,k)}|q_l,\boldsymbol{Z}_{-1,l}}(\xi|0, \boldsymbol{Z}_{-1,l})$*,* $J_{i,l}^{(j,k)} = \mathcal{J}_{i,l}^{(j,k)} / f_{q_l|\boldsymbol{Z}_{-1,l}}(0|\boldsymbol{Z}_{l,i}^{(j,k)})$ *with* $\mathcal{J}_{i,l}^{(j,k)} =$

$s_l^{(j)} \sum_{n=1}^{i} \mathcal{E}_{l,n}^{(j,k)}$ and $\{\mathcal{E}_{l,n}^{(j,k)}\}_{n=1}^{\infty}$ are independent unit exponential variables which are independent of $\{(\xi_i^{(j,k)}, \boldsymbol{Z}_{l,i}^{(j,k)})\}_{i=1}^{\infty}$. Moreover, $\{(\xi_i^{(j,k)}, \boldsymbol{Z}_{l,i}^{(j,k)}, J_{l,i}^{(j,k)})\}_{i=1}^{\infty}$ are independent across $l = 1, 2$ and $(j,k) \in \mathcal{S}(l)$.

The following Lemma B.3 establishes the stochastic equi-lower-semicontinuity of $\{D_T(\boldsymbol{v})\}$, which together with the finite-dimensional converges in distribution implies the epi-convergence in distribution.

LEMMA B.3. *Under Assumptions 1-5, the sequence $\{D_T(\boldsymbol{v})\}$ defined in (B.25) is stochastic equi-lower-semicontinuous, namely that for any compact set $B \subset \mathbb{R}^{d_1+d_2}$ and any $\epsilon, \delta > 0$, there exists $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m \in B$, where m is a finite integer depending on B, and some open sets $V(\boldsymbol{v}_1), \cdots, V(\boldsymbol{v}_m)$ covering B and containing $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m$, such that*

$$\limsup_{T\to\infty} \mathbb{P}\left(\cup_{j=1}^{m} \left\{ \inf_{\boldsymbol{v}\in V(\boldsymbol{v}_j)} D_T(\boldsymbol{v}) \le \min(\epsilon^{-1}, D_T(\boldsymbol{v}_j) - \epsilon) \right\} \right) < \delta.$$

To present our next lemma, we first define the following class of piece-wise constant functions on $\mathbb{R}^d$ as

$$\mathcal{F} = \left\{ f(\boldsymbol{v}) = \sum_{i=0}^{\infty} a_i \mathbb{1}\{\boldsymbol{v} \in F_i\}, \;\; a_i \in \mathbb{R}, F_i \text{ is a connected set in } \mathbb{R}^d, F_i \cap F_j = \emptyset \text{ if } i \neq j \right\}.$$

For each $f \in \mathcal{F}$, let $\tilde{f} = \sum_{i=0}^{\infty} i \mathbb{1}\{\boldsymbol{v} \in F_i\}$ be its associated pure jump process, which has a jump size 1 when moving from F_i to F_{i+1}. We refer to the sets $\{F_i\}$ as the level sets for f and $\tilde{f}$. Note that any realization of both $D_T(\boldsymbol{v})$ and $D(\boldsymbol{v})$ belongs to $\mathcal{F}$. Lemma B.4 below ensures that the centroid of the armgin set of $f \in \mathcal{F}$, when viewed as a functional from $\mathcal{F}$ to $\mathbb{R}$, is a continuous mapping functional under the topology of epi-convergence. It is similar in spirit to Lemma 3.1 of Lan et al. (2009), where they established the smallest and largest argmin functionals are continuous mappings in the univariate Skorohod space, while our result is under the metric induced by the epi-convergence in multivariate space.

LEMMA B.4. *Given a compact space E for $\boldsymbol{v}$, suppose that (i) on the domain E, the sequence $\{f_n \in \mathcal{F}\}$ epi-converges to $f_0 \in \mathcal{F}$ and its jump process $\{\tilde{f}_n\}$ also epi-converges to $\tilde{f}_0$; (ii) there are finite numbers of jumps of $\{\tilde{f}_n\}$ and $\tilde{f}_0$ in E; (iii) f_0 has a unique level set. Let G_n and G_0 be the set in E on which f_n and f_0 are minimized, respectively. Then,*

$$\frac{\int \boldsymbol{v}\mathbb{1}(\boldsymbol{v} \in G_n) d\boldsymbol{v}}{\int \mathbb{1}(\boldsymbol{v} \in G_n) d\boldsymbol{v}} \to \frac{\int \boldsymbol{v}\mathbb{1}(\boldsymbol{v} \in G_0) d\boldsymbol{v}}{\int \mathbb{1}(\boldsymbol{v} \in G_0) d\boldsymbol{v}}, \quad \text{as } n \to \infty. \tag{B.27}$$

Let $\ell^\infty(\mathbb{B})$ be the space of all bounded functions equipped with the uniform norm on the domain $\mathbb{B}$, where $\mathbb{B}$ is the parameter space for $\boldsymbol{\beta}$. The following lemma establishes the weak convergence of W_T in $\ell^\infty(\mathbb{B})$ and its asymptotic independence with D_T.

LEMMA B.5. *Under Assumptions 1-5, the sequence $\{W_T\}_{T=1}^{\infty}$ defined in (B.24) weakly converges to W in $\ell^\infty(\mathbb{B})$, where for any $\boldsymbol{u} = (\boldsymbol{u}_1^{\mathrm{T}}, \cdots, \boldsymbol{u}_4^{\mathrm{T}})^{\mathrm{T}}$, $W(\boldsymbol{u}) = \sum_{k=1}^{4} W_k(\boldsymbol{u}_k)$,*

$$W_k(\boldsymbol{u}_k) = \boldsymbol{u}_k^{\mathrm{T}} \mathbb{E}\left[\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}} \mathbb{1}\left\{\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0)\right\}\right] \boldsymbol{u}_k - 2\boldsymbol{u}_k^{\mathrm{T}} \boldsymbol{H}_k, \tag{B.28}$$

$\boldsymbol{H}_k \sim \boldsymbol{N}(\boldsymbol{0}, \Sigma_k)$ and $\Sigma_k = \mathbb{E}\left[\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}} \varepsilon^2 \mathbb{1}\left\{\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0)\right\}\right]$. Furthermore, the random function $W(\boldsymbol{u})$ is independent of $D(\boldsymbol{v})$ defined in (B.26).

With the above Lemmas B.1–B.5, we are now ready to prove Theorem 3.3 as follows.

Proof of Theorem 3.3

PROOF. Let $\boldsymbol{V}_T = T(\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0)$ with $\widehat{\boldsymbol{\gamma}} \in \widehat{G}$ and $\boldsymbol{U}_T = \sqrt{T}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)$ be standardizations of the LSEs for $\boldsymbol{\gamma}_0$ and $\boldsymbol{\beta}_0$, respectively. By the definition of $(\widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\beta}})$,

$$(\boldsymbol{V}_T, \boldsymbol{U}_T) \in \underset{(\boldsymbol{v},\boldsymbol{u})}{\arg\min} \left[T\mathbb{E}_T \left\{ m(\boldsymbol{W}_t, \boldsymbol{\beta}_0 + \frac{\boldsymbol{u}}{\sqrt{T}}, \boldsymbol{\gamma}_0 + \frac{\boldsymbol{v}}{T}) - m(\boldsymbol{W}_t, \boldsymbol{\beta}_0, \boldsymbol{\gamma}_0) \right\} \right]$$
$$\in \underset{(\boldsymbol{v},\boldsymbol{u})}{\arg\min}\, Q_T(\boldsymbol{v}, \boldsymbol{u}), \tag{B.29}$$

where $\mathbb{E}_T$ is the empirical average operator, $Q_T(\boldsymbol{v}, \boldsymbol{u})$ is defined in (B.22), $\boldsymbol{v} = (\boldsymbol{v}_1^{\mathrm{T}}, \boldsymbol{v}_2^{\mathrm{T}})^{\mathrm{T}}$ and $\boldsymbol{u} = (\boldsymbol{u}_1^{\mathrm{T}}, \cdots, \boldsymbol{u}_4^{\mathrm{T}})^{\mathrm{T}}$. The proof includes the following three steps: (1) the separability and finite-dimensional convergence of $\{Q_T(\boldsymbol{v}, \boldsymbol{u})\}_{T=1}^{\infty}$, (2) the epi-convergence of the random functions $\{Q_T\}_{T=1}^{\infty}$ to Q, and (3) the continuous mapping for the centroid of the argmin set.

Step 1. Separability and finite-dimensional convergence.

According to Lemma B.1, $Q_T(\boldsymbol{v}, \boldsymbol{u})$ can be separated as

$$Q_T(\boldsymbol{v}, \boldsymbol{u}) = W_T(\boldsymbol{u}) + D_T(\boldsymbol{v}) + o_p(1), \tag{B.30}$$

uniformly for $(\boldsymbol{u}^{\mathrm{T}}, \boldsymbol{v}^{\mathrm{T}})^{\mathrm{T}}$ in any compact set of $\mathbb{R}^{4p+d_1+d_2}$, where $W_T(\boldsymbol{u})$ and $D_T(\boldsymbol{v})$ are defined in (B.24) and (B.25), respectively.

Let $Q(\boldsymbol{v}, \boldsymbol{u}) = W(\boldsymbol{u}) + D(\boldsymbol{v})$, where $W(\boldsymbol{u})$ is defined in (B.24) and $D(\boldsymbol{v})$ is given in (B.26). Note that $D(\boldsymbol{v}) = D_1(\boldsymbol{v}_1) + D_2(\boldsymbol{v}_2)$, where

$$D_l(\boldsymbol{v}_l) = \sum_{j,k \in \mathcal{S}(l)} \sum_{i=1}^{\infty} \xi_i^{(j,k)} \mathbb{1} \left\{ s_l^{(j)} \left(J_{i,l}^{(j,k)} + (\boldsymbol{Z}_{l,i}^{(j,k)})^{\mathrm{T}} \boldsymbol{v}_{-1,l} \right) \le 0 < s_l^{(j)} J_{i,l}^{(j,k)} \right\},$$

for $l = 1$ and 2. By Lemma B.2, for any finite positive integer k and $(\boldsymbol{v}_{(1)}, \cdots, \boldsymbol{v}_{(k)})$ where $\boldsymbol{v}_{(i)} = (\boldsymbol{v}_{(i),1}^{\mathrm{T}}, \boldsymbol{v}_{(i),2}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{d_1+d_2}$ for each $i \in \{1, \cdots, 4\}$, we have

$$\left(D_T(\boldsymbol{v}_{(1)}), \cdots, D_T(\boldsymbol{v}_{(k)}) \right) \xrightarrow{d} \left(D(\boldsymbol{v}_{(1)}), \cdots, D(\boldsymbol{v}_{(k)}) \right), \tag{B.31}$$

namely, $D(\boldsymbol{v})$ is the finite-dimensional limiting distribution of $D_T(\boldsymbol{v})$. The finite-dimensional weak convergence of $W_T(\boldsymbol{u})$ to $W(\boldsymbol{u})$ is implied by Lemma B.5. Therefore, $Q_T(\boldsymbol{u}, \boldsymbol{v})$ weakly converges to $Q(\boldsymbol{u}, \boldsymbol{v})$ in the finite-dimensional sense.

Step 2. Epi-convergence.

Lemma B.3 establishes the stochastic equi-lower-semicontinuouity (s.e-l-sc) of the sequence $\{D_T\}_{T=1}^{\infty}$. From the regular form of $\{W_T\}_{T=1}^{\infty}$, this sequence of random functions converges in distribution to W with respect to the topology of uniform convergence, implying $\{W_T\}_{T=1}^{\infty}$ epi-converge in distribution to W. Then by the finite-dimensional convergence of $\{W_T\}_{T=1}^{\infty}$ implied from Lemma B.5 and Theorem 3 of Knight (1999), $\{W_T\}_{T=1}^{\infty}$ is a sequence of s.e-l-sc random functions. Consequently, $\{Q_T\}_{T=1}^{\infty}$ are s.e-l-sc, which together with the finite-dimensional weak convergence shown in Step 1 implies that $\{Q_T\}_{T=1}^{\infty}$ epi-converges in distribution to Q by Lemma A.9.

For any given $\boldsymbol{v}$, by the separatability of $Q(\boldsymbol{u},\boldsymbol{v}) = W(\boldsymbol{u}) + D(\boldsymbol{v})$, where $W(\boldsymbol{u})$ is quadratic in $\boldsymbol{u}$ as shown in (B.28), we can see that $Q(\boldsymbol{u},\boldsymbol{v})$ is minimized at $U = (U_1,\cdots,U_4)^{\tau}$, where for $k \in \{1,\cdots,4\}$,

$$U_k = \mathbb{E}\left[\boldsymbol{X}\boldsymbol{X}^{\tau}\mathbb{1}\left\{\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0)\right\}\right]^{-1} H_k, \quad H_k \sim \boldsymbol{N}(\boldsymbol{0},\Sigma_k),$$

and Σ_k is given in Lemma B.5. By Theorem 1 of Knight (1999), we obtain $\sqrt{T}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) = U_T \xrightarrow{d} U$. Let G_D be the argmin set of $D(\boldsymbol{v})$. Since Assumption 3.(ii) implies that neither $\boldsymbol{Z}_{1,t}$ nor $\boldsymbol{Z}_{2,t}$ is multicollinear, following the same arguments as in Yu and Fan (2021), it can be shown G_D is compact almost surely, so that its centroid is well defined. It is worth noting that because the minimizers of $D(\boldsymbol{v})$ are not unique, Theorem 1 of Knight (1999) can not be directly applicable to imply the weak convergence of $\arg\min_{\boldsymbol{v}} D_T(\boldsymbol{v})$ to that of $\arg\min_{\boldsymbol{v}} D(\boldsymbol{v})$. Instead, we consider the centroid of argmin, which can be viewed as a continuous functional of a process, to obtain the desired weak convergence in Theorem 3.3.

Step 3. Continuous mapping for the centroid of the argmin set.

Since $\{D_T(\boldsymbol{v})\}_{T=1}^{\infty}$ and $D(\boldsymbol{v})$ can be endowed into a complete and separable metric space induced by the epi-convergence, we can find a probability space and random elements with $D'_T(\boldsymbol{v}) \overset{d}{=} D_T(\boldsymbol{v})$ for each $T \geq 1$ and $D'(\boldsymbol{v}) \overset{d}{=} D(\boldsymbol{v})$, such that $D'_T(\boldsymbol{v})$ epi-converges to $D'(\boldsymbol{v})$ with probability 1 (van der Vaart and Wellner, 1996). Let $\widehat{\mathcal{G}}'$ and $\mathcal{G}'_D$ be the argmin sets of $D'_T(\boldsymbol{v})$ and $D'(\boldsymbol{v})$, respectively. Condition (i) of Lemma B.4 is ensured by the epi-convergence of $\{D'_T(\boldsymbol{v})\}$ to $D'(\boldsymbol{v})$. Because the point process induced by $\{Tq_{l,t}\}$ is asymptotic Poisson, there are stochastically finite number of jumps in any compact region, and Condition (ii) Lemma B.4 holds with the probability approaching 1. Also, Condition (iii) is ensured by the continuity of the jump size $\xi_i^{(j,k)}$ of $D(\boldsymbol{v})$. Applying Lemma B.4, we have $C(\widehat{\mathcal{G}}') \to C(\mathcal{G}'_D)$, where $C(E)$ denotes the centroid of any bounded set E. Hence, we conclude that $T(\widehat{\boldsymbol{\gamma}}^c - \boldsymbol{\gamma}_0) = C(\widehat{\mathcal{G}}) \xrightarrow{d} C(\mathcal{G}_D) = \boldsymbol{\gamma}_D^c$. Finally, the asymptotic independence between $\sqrt{T}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)$ and $T(\widehat{\boldsymbol{\gamma}}^c - \boldsymbol{\gamma}_0)$ is implied by the independence between $W(\boldsymbol{u})$ and $D(\boldsymbol{v})$ established in Lemma B.5. Because $T(\widehat{\boldsymbol{\gamma}}_1^c - \boldsymbol{\gamma}_{10})$ and $T(\widehat{\boldsymbol{\gamma}}_2^c - \boldsymbol{\gamma}_{20})$ depend asymptotically on $D_1(\boldsymbol{v})$ and $D_2(\boldsymbol{v})$, respectively, which are shown to be independent in Part 3 of the proof of Lemma B.2, the asymptotic independence between $T(\widehat{\boldsymbol{\gamma}}_1^c - \boldsymbol{\gamma}_{10})$ and $T(\widehat{\boldsymbol{\gamma}}_2^c - \boldsymbol{\gamma}_{20})$ follows. □

B.6. Proof of Lemma B.1.

PROOF. First, the left-hand of (B.23) admits the following decomposition:

$$\begin{aligned}
&T\mathbb{E}_T\{m(\boldsymbol{W}_t,\boldsymbol{\beta}_0 + \frac{\boldsymbol{u}}{\sqrt{T}},\boldsymbol{\gamma}_0 + \frac{\boldsymbol{v}}{T}) - m(\boldsymbol{W}_t,\boldsymbol{\beta}_0,\boldsymbol{\gamma}_0)\} \\
=&\sum_{j=1}^{4}\sum_{t=1}^{T}(\boldsymbol{u}_j^{\tau}\frac{\boldsymbol{X}_t\boldsymbol{X}_t^{\tau}}{T}\boldsymbol{u}_j - \boldsymbol{u}_j^{\tau}\frac{2}{\sqrt{T}}\boldsymbol{X}_t\varepsilon_t)\mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0 + \frac{\boldsymbol{v}}{T}) \\
+&\sum_{i\neq j}^{4}\sum_{t=1}^{T}\{(\boldsymbol{\delta}_{ij,0} - \frac{\boldsymbol{u}_j}{\sqrt{T}})^{\tau}\boldsymbol{X}_t\boldsymbol{X}_t^{\tau}(\boldsymbol{\delta}_{ij,0} - \frac{\boldsymbol{u}_j}{\sqrt{T}}) + 2\boldsymbol{X}_t^{\tau}(\boldsymbol{\delta}_{ij,0} - \frac{\boldsymbol{u}_j}{\sqrt{T}})\varepsilon_t\}\mathbb{1}_t^{(i)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0 + \frac{\boldsymbol{v}}{T}) \\
=&\sum_{j=1}^{4}H_j(\boldsymbol{h}) + \sum_{i\neq j}^{4}F_{ij}(\boldsymbol{h}), \quad \text{say.}
\end{aligned} \tag{B.32}$$

For the H_j term, let $R_{j,t} = \boldsymbol{u}_j^{\mathrm{T}} \boldsymbol{X}_t \boldsymbol{X}_t^{\mathrm{T}} \boldsymbol{u}_j \mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)$, by the ULLN in Lemma A.1,

$$(\mathbb{E}_T - \mathbb{E})\{R_{j,t}\mathbb{1}_t^{(j)}\left(\boldsymbol{\gamma}_0 + \frac{\boldsymbol{v}}{T}\right)\} = o_p(1). \tag{B.33}$$

Note that

$$\begin{aligned}
&\mathbb{E}\{\left|R_{j,t}\mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0 + \frac{\boldsymbol{v}}{T}) - R_{j,t}\right|\} \\
\leq &\sum_{l=1}^{2} \mathbb{E}\{R_{j,t}|\mathbb{1}_{l,t}(\boldsymbol{\gamma}_{l0}) - \mathbb{1}_{l,t}(\boldsymbol{\gamma}_{l0} + \frac{\boldsymbol{v}_l}{T})|\} \overset{(i)}{\lesssim} \frac{\sum_{l=1}^{2}\|\boldsymbol{v}_l\|}{T} = o(1),
\end{aligned} \tag{B.34}$$

where (i) is implied by Lemma A.2. Then, combining (B.33) and (B.34) yields

$$\begin{aligned}
&\sum_{t=1}^{T} \boldsymbol{u}_j^{\mathrm{T}} \frac{\boldsymbol{X}_t \boldsymbol{X}_t^{\mathrm{T}}}{T} \boldsymbol{u}_j \mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}\left(\boldsymbol{\gamma}_0 + \frac{\boldsymbol{v}}{T}\right) \\
=&\mathbb{E}_T\left\{R_{j,t}\mathbb{1}_t^{(j)}\left(\boldsymbol{\gamma}_0 + \frac{\boldsymbol{v}}{T}\right)\right\} \\
=&\mathbb{E}(R_{j,t}) + o_p(1) = \sum_{j=1}^{4} \boldsymbol{u}_j^{\mathrm{T}} \mathbb{E}\left\{\boldsymbol{X}_t \boldsymbol{X}_t^{\mathrm{T}} \mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)\right\} \boldsymbol{u}_j + o_p(1).
\end{aligned} \tag{B.35}$$

For the second part of $H_j(\boldsymbol{h})$, let $S_{j,t} = 2\boldsymbol{u}_j^{\mathrm{T}} \boldsymbol{X}_t \varepsilon_t \mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)$. Note that

$$\begin{aligned}
&\sqrt{T}\mathbb{E}_T\left[S_{j,t}\left\{\mathbb{1}_t^{(j)}\left(\boldsymbol{\gamma}_0 + \frac{\boldsymbol{v}}{T}\right) - \mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)\right\}\right] \\
\leq &\sum_{l=1}^{2} \sqrt{T}\mathbb{E}_T\left\{|S_{j,t}|\left|\mathbb{1}_{l,t}(\boldsymbol{\gamma}_{l0}) - \mathbb{1}_{l,t}(\boldsymbol{\gamma}_{l0} + \frac{\boldsymbol{v}_l}{T})\right|\right\} = o_p(1),
\end{aligned}$$

according to (A.20) in Lemma A.5. Hence, applying Lemma A.5 gives

$$\sqrt{T}\mathbb{E}_T\left\{S_{j,t}\mathbb{1}_t^{(j)}\left(\boldsymbol{\gamma}_0 + \frac{\boldsymbol{v}}{T}\right)\right\} = \sqrt{T}\mathbb{E}_T\left\{S_{j,t}\mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)\right\} + o_p(1). \tag{B.36}$$

Combining (B.35) and (B.36) and summing across $j = 1, \cdots, 4$ leads to

$$\sum_{j=1}^{4} H_j(\boldsymbol{h}) = W_T(\boldsymbol{u}) + o_p(1). \tag{B.37}$$

For the $F_{ij}(\boldsymbol{h})$ terms ($i \neq j \in \{1, \cdots, 4\}$) in (B.32), we divide them into two cases according to whether there exists $l \in \{1, 2\}$ such that $(i, j) \in \mathcal{S}(\boldsymbol{\gamma}_{l0})$ or not. For those (i, j) that does not have $l \in \{1, 2\}$ such that $(i, j) \in \mathcal{S}(l)$, i.e., $s_1^{(i)} \neq s_1^{(j)}$ and $s_2^{(i)} \neq s_2^{(j)}$,

$$\mathbb{1}_t^{(i)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}\left(\boldsymbol{\gamma}_0 + \frac{\boldsymbol{v}}{T}\right) \leq \left|\mathbb{1}_{1,t}(\boldsymbol{\gamma}_{l0}) - \mathbb{1}_{1,t}\left(\boldsymbol{\gamma}_{10} + \frac{\boldsymbol{v}_1}{T}\right)\right| \left|\mathbb{1}_{2,t}(\boldsymbol{\gamma}_{h0}) - \mathbb{1}_{2,t}\left(\boldsymbol{\gamma}_{20} + \frac{\boldsymbol{v}_2}{T}\right)\right|.$$

Then, applying (A.21) in Lemma A.5, where we define U_t in Lemma A.5 as

$$\left|(\boldsymbol{\delta}_{ij,0} - \frac{\boldsymbol{u}_j}{\sqrt{T}})^{\mathrm{T}} \boldsymbol{X}_t \boldsymbol{X}_t^{\mathrm{T}} (\boldsymbol{\delta}_{ij,0} - \frac{\boldsymbol{u}_j}{\sqrt{T}}) + 2\boldsymbol{X}_t^{\mathrm{T}} (\boldsymbol{\delta}_{ij,0} - \frac{\boldsymbol{u}_j}{\sqrt{T}})\varepsilon_t\right|,$$

yields that

$$F_{ij}(\boldsymbol{h}) = o_p(1), \text{ if } (i, j) \notin \mathcal{S}(l) \text{ for any } l \in \{1, 2\}. \tag{B.38}$$

Otherwise, if there exists $l \in \{1, 2\}$ such that $(i, j) \in \mathcal{S}(l)$,

$$F_{ij}(\boldsymbol{h}) = T\mathbb{E}_T\{\xi_t^{(i,j)}\mathbb{1}_t^{(i)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0 + \frac{\boldsymbol{v}}{T})\} + \sqrt{T}\mathbb{E}_T\{T_t^{(i,j)}\mathbb{1}_t^{(i)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0 + \frac{\boldsymbol{v}}{T})\}$$

$$+\mathbb{E}_T\{U_t^{(i,j)}\mathbb{1}_t^{(i)}(\gamma_0)\mathbb{1}_t^{(j)}(\gamma_0+\frac{\boldsymbol{v}}{T})\}+\sqrt{T}\mathbb{E}_T\{V_t^{(i,j)}\mathbb{1}_t^{(i)}(\gamma_0)\mathbb{1}_t^{(j)}(\gamma_0+\frac{\boldsymbol{v}}{T})\}, \tag{B.39}$$

where $\xi_t^{(i,j)}$ is defined in (B.25), and

$$T_t^{(i,j)}=\boldsymbol{\delta}_{ij,0}^{\mathrm{T}}\boldsymbol{X}_t\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{u}_j,\ \ U_t^{(i,j)}=\boldsymbol{u}_j^{\mathrm{T}}\boldsymbol{X}_t\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{u}_j \text{ and } V_t^{(i,j)}=-2\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{u}_j\varepsilon_t.$$

For the first term on the right-hand side of (B.39), we note that $\mathbb{1}_t^{(i)}(\gamma_0)\mathbb{1}_t^{(j)}\left(\gamma_0+\frac{\boldsymbol{v}}{T}\right)=1$ means that $\boldsymbol{Z}_t$ is classified into $R_i(\gamma_0)$ under the true γ_0, but is classified into $R_j(\gamma)$ under the candidate parameter γ. Since the i-th and the j-th regions are on the opposite sides of the l-th hyperplane, while are on the same side of the h-th hyperplane for the $h\neq l\in\{1,2\}$, we have the following two implications: (i) $\mathrm{sign}(\boldsymbol{Z}_{l,t}^{\mathrm{T}}\gamma_{l0})\neq\mathrm{sign}\left\{\boldsymbol{Z}_{l,t}^{\mathrm{T}}\left(\gamma_{l0}+\frac{\boldsymbol{v}_l}{T}\right)\right\}$, which is equivalent to

$$\mathbb{1}\left\{s_l^{(i)}\boldsymbol{Z}_{l,t}^{\mathrm{T}}\left(\gamma_{l0}+\frac{\boldsymbol{v}_l}{T}\right)\leq 0<s_l^{(i)}\boldsymbol{Z}_{l,t}^{\mathrm{T}}\gamma_{l0}\right\}=1;$$

and (ii) $\mathrm{sign}(\boldsymbol{Z}_{h,t}^{\mathrm{T}}\gamma_{h0})=\mathrm{sign}\left\{\boldsymbol{Z}_{h,t}^{\mathrm{T}}\left(\gamma_{h0}+\frac{\boldsymbol{v}_h}{T}\right)\right\}$ for $h\neq l\in\{1,2\}$, which is equivalent to

$$\mathbb{1}\left\{0<\min\{s_h^{(i)}\boldsymbol{Z}_{h,t}^{\mathrm{T}}\gamma_{h0},s_h^{(i)}\boldsymbol{Z}_{h,t}^{\mathrm{T}}\left(\gamma_{h0}+\frac{\boldsymbol{v}_h}{T}\right)\}\right\}=1.$$

For $(i,j)\in\mathcal{S}(l)$, let $\mathbb{1}_t^{(i,j)}(\gamma_0)=\mathbb{1}_t^{(i)}(\gamma_0)+\mathbb{1}_t^{(j)}(\gamma_0)$. It is noted that

$$\begin{aligned}&\left|\mathbb{1}_t^{(i,j)}(\gamma_0)\mathbb{1}_t^{(j)}(\gamma_0+\frac{\boldsymbol{v}}{T})-\mathbb{1}_t^{(i,j)}(\gamma_0)\mathbb{1}_{l,t}\{s_l^{(i)}\boldsymbol{Z}_{l,t}^{\mathrm{T}}(\gamma_{l0}+\frac{\boldsymbol{v}_l}{T})\leq 0<s_l^{(i)}\boldsymbol{Z}_{l,t}^{\mathrm{T}}\gamma_{l0}\}\right|\\ \leq&\left|\mathbb{1}_{1,t}(\gamma_{10})-\mathbb{1}_{1,t}(\gamma_{10}+\frac{\boldsymbol{v}_1}{T})\right|\left|\mathbb{1}_{2,t}(\gamma_{20})-\mathbb{1}_{2,t}(\gamma_{20}+\frac{\boldsymbol{v}_2}{T})\right|.\end{aligned} \tag{B.40}$$

Applying (A.21) in Lemma A.5, we have

$$T\mathbb{E}_T\left\{|\xi_t^{(i,j)}|\left|\mathbb{1}_{1,t}(\gamma_{10})-\mathbb{1}_{1,t}\left(\gamma_{10}+\frac{\boldsymbol{v}_1}{T}\right)\right|\left|\mathbb{1}_{2,t}(\gamma_{20})-\mathbb{1}_{2,t}\left(\gamma_{20}+\frac{\boldsymbol{v}_2}{T}\right)\right|\right\}=o_p(1),$$

which, together with (B.40), implies that

$$\begin{aligned}&T\mathbb{E}_T\left\{\xi_t^{(i,j)}\mathbb{1}_t^{(i)}(\gamma_0)\mathbb{1}_t^{(j)}\left(\gamma_0+\frac{\boldsymbol{v}}{T}\right)\right\}\\ =&T\mathbb{E}_T\left\{\xi_t^{(i,j)}\mathbb{1}_{l,t}\left\{s_l^{(i)}\boldsymbol{Z}_{l,t}^{\mathrm{T}}\left(\gamma_{l0}+\frac{\boldsymbol{v}_l}{T}\right)\leq 0<s_l^{(i)}\boldsymbol{Z}_{l,t}^{\mathrm{T}}\gamma_{l0}\right\}\right\}+o_p(1)\\ =&T\mathbb{E}_T\left\{\xi_t^{(i,j)}\mathbb{1}_{l,t}\left\{s_l^{(i)}\left(Tq_{l,t}+\boldsymbol{Z}_{l,t}^{\mathrm{T}}\boldsymbol{v}_l\right)\leq 0<s_l^{(i)}Tq_{l,t}\right\}\right\}+o_p(1)\\ =&D_T^{(i,j)}(\boldsymbol{v})+o_p(1),\ \ \text{say},\end{aligned} \tag{B.41}$$

where in the second equality $q_{l,t}=\boldsymbol{Z}_{l,t}^{\mathrm{T}}\gamma_0$.

For the second term of (B.39), note that

$$\left|T_t^{(i,j)}\mathbb{1}_t^{(i)}(\gamma_0)\mathbb{1}_t^{(j)}\left(\gamma_0+\frac{\boldsymbol{v}}{T}\right)\right|\leq\left|T_t^{(i,j)}\right|\left|\mathbb{1}_{l,t}(\gamma_{l0})-\mathbb{1}_{l,t}(\gamma_{l0}+\frac{\boldsymbol{v}_l}{T})\right|.$$

According to (A.20) in Lemma A.5, it holds that

$$\sqrt{T}\mathbb{E}_T\left\{T_t^{(i,j)}\mathbb{1}_t^{(i)}(\gamma_0)\mathbb{1}_t^{(j)}\left(\gamma_0+\frac{\boldsymbol{v}}{T}\right)\right\}=o_p(1). \tag{B.42}$$

With the same arguments, we have

$$\mathbb{E}_T\left\{U_t^{(i,j)}\mathbb{1}_t^{(i)}(\gamma_0)\mathbb{1}_t^{(j)}\left(\gamma_0+\frac{\boldsymbol{v}}{T}\right)\right\}=o_p(1), \tag{B.43}$$

$$\sqrt{T}\mathbb{E}_T\left\{V_t^{(i,j)}\mathbb{1}_t^{(i)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}\left(\boldsymbol{\gamma}_0+\frac{\boldsymbol{v}}{T}\right)\right\}=o_p(1). \tag{B.44}$$

Finally, combining (B.39) and the four parts (B.41)–(B.44) yields

$$F_{ij}(\boldsymbol{h})=D_T^{(i,j)}(\boldsymbol{v})+o_p(1), \text{ if there exists } l\in\{1,2\} \text{ such that } (i,j)\in\mathcal{S}(l). \tag{B.45}$$

Since $Q_T(\boldsymbol{u},\boldsymbol{v})=\sum_{j=1}^4 H_j(\boldsymbol{h})+\sum_{i\neq j}^4 F_{ij}(\boldsymbol{h})$ as shown in (B.32), using (B.37) for the $H_j(\boldsymbol{h})$ terms, and (B.38) and (B.45) for the $F_{ij}(\boldsymbol{h})$ terms, the desired result (B.23) for the decomposition of $Q_T(\boldsymbol{u},\boldsymbol{v})$ is obtained. □

B.7. Proof of Lemma B.2.

PROOF. For notational simplicity, in this proof, we show the marginal weak convergence of $D_T(\boldsymbol{v})$, i.e., $D_T(\boldsymbol{v})\xrightarrow{d}D(\boldsymbol{v})$ for any fixed $\boldsymbol{v}$, since the finite-dimensional weak convergence can be easily extended with the similar argument but more involved notations. Specifically, to show that $\big(D_T(\boldsymbol{v}_{(1)}),\cdots,D_T(\boldsymbol{v}_{(m)})\big)\xrightarrow{d}\big(D(\boldsymbol{v}_{(1)}),\cdots,D(\boldsymbol{v}_{(m)})\big)$ for any finite integer m, it suffices to replace the mapping $\mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)}$ defined in (B.47) associated with the marginal $\boldsymbol{v}=(\boldsymbol{v}_1^{\mathrm{T}},\boldsymbol{v}_2^{\mathrm{T}})^{\mathrm{T}}$ to a m-dimensional mapping $(\mathcal{T}_{l,\boldsymbol{v}_{(1),l}}^{(j,k)},\cdots,\mathcal{T}_{l,\boldsymbol{v}_{(m),l}}^{(j,k)})$ for each $l\in\{1,2\}$ and $(j,k)\in\mathcal{S}(l)$.

The proof is divided to four parts. In Part 1 we express D_T as a functional of point processes. Part 2 first establishes the weak limit of the empirical point process, by verifying Meyer's condition which ensures the asymptotical Poisson for the point process with the mixing sequences. Then we construct an explicit representation of the limiting process. Part 3 shows the asymptotical independence of the point processes associated with different splitting hyperplanes. In Part 4, we employ a continuous mapping theorem for the functional introduced in Part 1 to obtain the weak convergence of $D_T(\boldsymbol{v})$.

Part 1: Transformation into a functional of point processes. In this part, we will express $D_T(\boldsymbol{v})$ as a sum of transformations of point processes.

Recall that

$$D_T(\boldsymbol{v})=\sum_{l=1}^{2}\sum_{t=1}^{T}\sum_{(j,k)\in\mathcal{S}(l)}\xi_t^{(j,k)}\mathbb{1}\left\{s_l^{(j)}\left(Tq_{l,t}+\boldsymbol{Z}_{-1,l,t}^{\mathrm{T}}\boldsymbol{v}_{-1,l}\right)\le 0<s_l^{(j)}Tq_{l,t}\right\},$$

where $\xi_t^{(j,k)}=\left(\boldsymbol{\delta}_{jk,0}^{\mathrm{T}}\boldsymbol{X}_t\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{jk,0}+2\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{jk,0}\varepsilon_t\right)\{\mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)+\mathbb{1}_t^{(k)}(\boldsymbol{\gamma}_0)\}$.

We now show that $D_T(\boldsymbol{v})$ can be written as a sum of functionals of some empirical point processes. For each $l\in\{1,2\}$ and $(j,k)\in\mathcal{S}(l)$, we define an empirical point process $\widehat{\mathbf{N}}_{l,T}^{(j,k)}\in M_p(E_l)$, which is the space of Radon point measures defined in Definition A.2, where $E_l=\mathbb{R}_{s_l^{(j)}}\times\mathcal{Z}_{-1,l}\times\mathbb{R}$, as

$$\widehat{\mathbf{N}}_{l,T}^{(j,k)}(F):=\sum_{t=1}^{T}\mathbb{1}\left\{(Tq_{l,t},\boldsymbol{Z}_{-1,l,t},\xi_t^{(j,k)})\in F\right\} \text{ for any} F=(F_1,F_2,F_3)\in E_l, \tag{B.46}$$

where $\mathbb{R}_{s_l^{(j)}}=(0,\infty)$ if $s_l^{(j)}=1$, and $\mathbb{R}_{s_l^{(j)}}=(-\infty,0]$ if $s_l^{(j)}=-1$. The element $\{0\}$ is excluded from the space of $\xi_t^{(j,k)}$ since $\xi_t^{(j,k)}=0$ does not affect $D_T(\boldsymbol{v})$.

For a given $\boldsymbol{v}=(\boldsymbol{v}_1^{\mathrm{T}},\boldsymbol{v}_2^{\mathrm{T}})^{\mathrm{T}}$, for each $l\in\{1,2\}$ and $(j,k)\in\mathcal{S}(l)$, we define a map $\mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)}:M_p(E_l)\to\mathbb{R}$ such that

$$\forall\,\mathbf{N}\in M_p(E):\ \mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)}(\mathbf{N})=\int_{E_l}g_{l,\boldsymbol{v}_l}^{(j,k)}(x,\boldsymbol{y},z)d\mathbf{N}(x,\boldsymbol{y},z), \tag{B.47}$$

where for each $x \in \mathbb{R}_{s_l^{(j)}}$, $\boldsymbol{y} \in \mathcal{Z}_{-1,l}$ and $z \in \mathbb{R}$,

$$g_{l,\boldsymbol{v}_l}^{(j,k)}(x, \boldsymbol{y}, z) = z \cdot \mathbb{1}\left\{ s_l^{(j)}(x + \boldsymbol{y}^{\mathrm{T}} \boldsymbol{v}_{-1,l}) \le 0 < s_l^{(j)} x \right\}.$$

Then, with (B.46) and (B.47) we can write

$$\sum_{t=1}^{T} \xi_t^{(j,k)} \mathbb{1}\left\{ s_l^{(j)} \left(T q_{l,t} + \boldsymbol{Z}_{-1,l,t}^{\mathrm{T}} \boldsymbol{v}_{-1,l} \right) \le 0 < s_l^{(j)} T q_{l,t} \right\} = \mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)} \left(\widehat{\mathbf{N}}_{l,T}^{(j,k)} \right).$$

Consequently, $D_T(\boldsymbol{v})$ can be expressed as

$$D_T(\boldsymbol{v}) = \sum_{l=1}^{2} \sum_{(j,k) \in \mathcal{S}(l)} \mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)} \left(\widehat{\mathbf{N}}_{l,T}^{(j,k)} \right). \tag{B.48}$$

Part 2: Weak limit of $\widehat{\mathbf{N}}_{l,T}^{(j,k)}$. In this part, we derive the weak limit of the empirical point process $\widehat{\mathbf{N}}_{l,T}^{(j,k)}$ for each $l \in L$ and $(j,k) \in \mathcal{S}_l$ in three steps. In step 1, we calculate $\lim_{T\to\infty} \mathbb{E}\left\{ \widehat{\mathbf{N}}_{l,T}^{(j,k)}(F) \right\}$ to obtain the mean measure of the limit process $\mathbf{N}_l^{(j,k)}$ required in (A.26) in Kallenberg's theorem (Lemma A.7). In the next step, we first verify Conditions (a)-(c) of Meyer's theorem (Lemma A.8), and then use it to show (A.27). The above two steps guarantee that the empirical point process $\widehat{\mathbf{N}}_{l,T}^{(j,k)}$ weakly converges to a Poisson process $\mathbf{N}_l^{(j,k)}$. In the final step, we will find an explicit representation of $\mathbf{N}_l^{(j,k)}$.

Step 1: Calculation of the limit of $\mathbb{E}\left\{ \widehat{\mathbf{N}}_{l,T}^{(j,k)}(F) \right\}$.

For any $F = (F_1, F_2, F_3) \in \mathcal{E}_l$, which is the basis of relatively compact open set in E_l, where $F_1 \subset \mathbb{R}_{s_l^{(j)}}$ and $F_2 \subset \mathcal{Z}_{-1,l}$, $F_3 \subset \mathbb{R}$, we have

$$\begin{aligned}
&\lim_{T\to\infty} \mathbb{E}\left\{ \widehat{\mathbf{N}}_{l,T}^{(j,k)}(F) \right\} \\
&= \lim_{T\to\infty} T \mathbb{P}\left\{ \left(T q_{l,t}, \boldsymbol{Z}_{-1,l,t}, \xi_t^{(j,k)} \right) \in F \right\} \\
&= \lim_{T\to\infty} T \int_{Tq\in F_1, \boldsymbol{z}\in F_2, \xi\in F_3} f_{\xi^{(j,k)}|(q_l, \boldsymbol{Z}_{-1,l})}(\xi | q, \boldsymbol{z}) f_{q_l|\boldsymbol{Z}_{-1,l}}(q|\boldsymbol{z}) f_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z}) \mathrm{d}q \mathrm{d}\boldsymbol{z} \mathrm{d}\xi \\
&\overset{(i)}{=} \lim_{T\to\infty} \int_{\tilde{q}\in F_1, \boldsymbol{z}\in F_2, \xi\in F_3} f_{\xi^{(j,k)}|(q_l, \boldsymbol{Z}_{-1,l})}(\xi \mid \frac{\tilde{q}}{T}, \boldsymbol{z}) f_{q_l|\boldsymbol{Z}_{-1,l}}(\frac{\tilde{q}}{T} \mid \boldsymbol{z}) f_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z}) \mathrm{d}\tilde{q} \mathrm{d}\boldsymbol{z} \mathrm{d}\xi \\
&\overset{(ii)}{=} \int_{\tilde{q}\in F_1, \boldsymbol{z}\in F_2, \xi\in F_3} f_{\xi^{(j,k)}|(q_l, \boldsymbol{Z}_{-1,l})}(\xi \mid 0, \boldsymbol{z}) f_{q_l|\boldsymbol{Z}_{-1,l}}(0 \mid \boldsymbol{z}) f_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z}) \mathrm{d}\tilde{q} \mathrm{d}\boldsymbol{z} \mathrm{d}\xi \\
&=: \mu_l^{(j,k)}(F) < \infty,
\end{aligned} \tag{B.49}$$

where (i) is by letting $q = \tilde{q}/T$, (ii) is by the dominating convergence theorem and the continuity of $f_{q_l|\boldsymbol{Z}_{-1,l}}(q|\boldsymbol{z})$ and $f_{\xi^{(j,k)}|(q_l, \boldsymbol{Z}_{-1,l})}(\xi \mid q, \boldsymbol{z})$ at $q = 0$, and that $\mu_l^{(j,k)}(F) < \infty$ is because of the uniform boundedness of the density functions assumed in Assumption 5 and the compactness of F. The measure $\mu_l^{(j,k)}$ on $E_l = \mathbb{R}_{s_l^{(j)}} \times \mathcal{Z}_{-1,l} \times \mathbb{R}$ is defined as

$$\mu_l^{(j,k)}(\mathrm{d}q, \mathrm{d}\boldsymbol{z}, \mathrm{d}\xi) = f_{\xi^{(j,k)}|(q_l, \boldsymbol{Z}_{-1,l})}(\xi \mid 0, \boldsymbol{z}) f_{q_l|\boldsymbol{Z}_{-1,l}}(0 \mid \boldsymbol{z}) f_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z}) \mathrm{d}q \mathrm{d}\boldsymbol{z} \mathrm{d}\xi. \tag{B.50}$$

Suppose that $\mu_l^{(j,k)}$ defined above is the mean measure of the point process $\mathbf{N}_l^{(j,k)}$, then (B.49) verifies the condition (A.26) required in Lemma A.7. To verify the other condition (A.27), we use Meyer's theorem, whose requirements are listed in (a)-(c) in Lemma A.8 and are verified as follows.

Step 2: Verification of the conditions of Meyer's theorem.

To show $\lim_{T\to\infty}\mathbb{P}\left\{\widehat{\mathbf{N}}_{l,T}^{(j,k)}(F)=0\right\}=\mathbb{P}\left\{\widehat{\mathbf{N}}_{l}^{(j,k)}(F)=0\right\}$, we now employ the Meyer's theorem presented in Lemma A.8. The following notations are the same as used in Lemma A.8. For any $F=(F_1,F_2,F_3)\in E_l$ and any sample size $n\geq 1$, define the sequence of "rare" events as

$$A_t^n(F)=\mathbb{1}\left\{(nq_{l,t},\boldsymbol{Z}_{-1,l,t},\xi_t^{(j,k)})\in F\right\},$$

for $1\leq t\leq n(n=1,2,\cdots)$. For any $m>0$, we take $q_m=[Lm]^q$ and $p_m=[Lm]^p$ for some $L\geq 1$ and $p\geq q\geq 1$, where $[x]$ denotes the largest interger not greater than x. Then $t_m=m(q_m+p_m)=m\left([Lm]^q+[Lm]^p\right)$. We illustrate the validity of Conditions (a)-(c) of Lemma A.8 as follows:

It is noted that Condition (a) is ensured by the condition of geometrical decaying α-mixing coefficient imposed in Assumption 1. Furthermore, Condition (b) is valid, since $q_m=[Lm]^q$ and $p_m=[Lm]^p$ for some constants $L\geq 1$ and $p\geq q>1$, leading to $p_{m+1}/p_m\to 1$ and $q_m/p_m\to 0$ as $m\to\infty$. Finally, for Condition (c), we note that

$$\begin{aligned}
t_m^2 I_{p_m}&=t_m^2\sum_{i=1}^{p_m-i}(p_m-i)\mathbb{P}\left\{A_1^{t_m}(F)\cap A_{i+1}^{t_m}(F)\right\}\\
&\leq t_m^2 p_m\sum_{i=1}^{p_m-i}\mathbb{P}\left\{A_1^{t_m}(F)\cap A_{i+1}^{t_m}(F)\right\}\\
&\leq t_m^2 p_m\sum_{i=1}^{p_m-i}\mathbb{P}\left\{(t_m q_{l,1}\in F_1)\cap(t_m q_{l,i+1}\in F_1)\right\}\\
&\overset{(iii)}{\lesssim} t_m^2 p_m^2\left\{\mathbb{P}\left(t_m q_{l,1}\in F_1\right)\right\}^2\\
&=t_m^2 p_m^2\left(\int_{t_m q\in F_1}dF_{q_l}(q)\right)^2\\
&\overset{(iv)}{=} t_m^2 p_m^2\left(\int_{t_m q\in F_1,\boldsymbol{z}\in\mathcal{Z}_{-1,l}}f_{q_l|\boldsymbol{Z}_{-1,l}}(q|\boldsymbol{z})f_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z})dqd\boldsymbol{z}\right)^2\\
&\overset{(v)}{=} p_m^2\left(\int_{q\in F_1,\boldsymbol{z}\in\mathcal{Z}_{-1,l}}f_{q_l|\boldsymbol{Z}_{-1,l}}(0|\boldsymbol{z})f_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z})dqd\boldsymbol{z}+o(1)\right)^2\overset{(vi)}{\leq} Cp_m^2
\end{aligned}\tag{B.51}$$

for some positive constant C, where $F_{q_l}(q)$ is the distribution function of $q_l=\boldsymbol{Z}_l^{\mathrm{T}}\boldsymbol{\gamma}_{l0}$. In the above derivation, (iii) is from Assumption 5 (i), (iv) is by conditioning $q_{l,1}$ on $\boldsymbol{Z}_{-1,l}$, (v) is obtained via the same arguments of (i) and (ii) used in deriving (B.49), and (vi) is because $f_{q_l|\boldsymbol{Z}_{-1,l}}$ is bounded with probability 1 by Assumption 5 (ii) and the compactness of F_1. Consequently, (B.51) implies that as $m\to\infty$,

$$I_{p_m}\leq C\frac{p_m^2}{t_m^2}=C\frac{p_m^2}{m^2(p_m+q_m)^2}\leq C\frac{1}{m^2}=o\left(\frac{1}{m}\right),$$

which verifies Condition (c) in Lemma A.8.

With Conditions (a)-(c) verified and $\mathbb{P}(A_t^T(F)) = \mu_l^{(j,k)}(F)/T + o(1/T)$ as shown in deriving (B.49), for any F with $\mu_l^{(j,k)}(F) > 0$, Meyer's theorem implies that

$$\begin{aligned}\lim_{T\to\infty} \mathbb{P}\left\{\widehat{\mathbf{N}}_{l,T}^{(j,k)}(F) = 0\right\} &= \lim_{T\to\infty} \mathbb{P}\left\{\text{none of } \left\{A_t^T(F)\right\}_{t=1}^T \text{ occurs}\right\} \\ &= e^{-\mu_l^{(j,k)}(F)} = \mathbb{P}\left\{\mathbf{N}_l^{(j,k)}(F) = 0\right\}, \end{aligned} \tag{B.52}$$

where $\mathbf{N}_l^{(j,k)}$ is a Poisson process with mean measure $\mu_l^{(j,k)}$. For F with $\mu_l^{(j,k)}(F) = 0$, (B.52) also holds, since in such case (B.49) implies $\mathbb{E}\left\{\widehat{\mathbf{N}}_{l,T}^{(j,k)}(F)\right\} \to 0$ as $T \to \infty$, which further implies that $\mathbb{P}\left\{\widehat{\mathbf{N}}_{l,T}(F) = 0\right\} = 1 = e^{-\mu_l^{(j,k)}(F)} = \mathbb{P}\left\{\mathbf{N}_l^{(j,k)}(F) = 0\right\}$. With (B.52) and (B.49), Kallenberg's theorem (Lemma A.7) implies that for each $l \in \{1, 2\}$ and $(j, k) \in \mathcal{S}(l)$, $\widehat{\mathbf{N}}_{l,T}^{(j,k)} \Rightarrow \mathbf{N}_l^{(j,k)}$ in $M_p(E_l)$ as $T \to \infty$.

Step 3. Representation of $\mathbf{N}_l^{(j,k)}$.

In this step, we construct a representation of $\mathbf{N}_l^{(j,k)}$ by applying the marking theorem (Proposition 3.8 of Resnick, 2008) twice. First, let $\mathbf{N}_{1,l}^{(j,k)}$ be a canonical Poisson process on $\mathbb{R}_{s_l^{(j)}}$ on points $\{\mathcal{J}_{l,i}^{(j,k)}\}_{i=1}^\infty$ defined as

$$\mathbf{N}_{1,l}^{(j,k)}(\cdot) = \sum_{i=1}^\infty \mathbb{1}\left\{\mathcal{J}_{l,i}^{(j,k)} \in \cdot\right\}, \quad \mathcal{J}_{l,i}^{(j,k)} = s_l^{(j)} \sum_{n=1}^i \mathcal{E}_{l,n}^{(j,k)}, \tag{B.53}$$

where $\left\{\mathcal{E}_{l,n}^{(j,k)}\right\}_{n=1}^\infty$ is an i.i.d. sequence of unit-exponential variables. Then $\mathbf{N}_{1,l}^{(j,k)}$ has the mean measure $\mu_{1,l}^{(j,k)}(dq) = dq$ on $\mathbb{R}_{s_l^{(j)}}$. Let $\{\boldsymbol{Z}_{l,i}^{(j,k)}\}_{i=1}^\infty$ be an i.i.d. sequence which follows the distribution $F_{\boldsymbol{Z}_{-1,l}}$ and is independent of $\left\{\mathcal{E}_{l,n}^{(j,k)}\right\}_{n=1}^\infty$. Then the marking theorem implies the composed process

$$\mathbf{N}_{2,l}^{(j,k)}(\cdot) = \sum_{i=1}^\infty \mathbb{1}\left\{\left(\mathcal{J}_{l,i}^{(j,k)}, \boldsymbol{Z}_{l,i}^{(j,k)}\right) \in \cdot\right\}$$

is a Poisson process with the mean measure $\mu_{2,l}^{(j,k)}(dq, d\boldsymbol{z}) = dq \cdot f_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z})d\boldsymbol{z}$ on $\mathbb{R}_{s_l^{(j)}} \times \mathcal{Z}_{-1,l}$. Let $\mathcal{T}_l : (q, \boldsymbol{z}) \to (q/f_{q|\boldsymbol{Z}_{-1,l}}(0|\boldsymbol{z}), \boldsymbol{z})$. Then by Proposition 3.7 in Resnick (2008),

$$\mathbf{N}_{3,l}^{(j,k)}(\cdot) = \sum_{i=1}^\infty \mathbb{1}\left\{\mathcal{T}_l\left(\mathcal{J}_{l,i}^{(j,k)}, \boldsymbol{Z}_{l,i}^{(j,k)}\right) \in \cdot\right\} = \sum_{i=1}^\infty \mathbb{1}\left\{\left(\frac{\mathcal{J}_{l,i}^{(j,k)}}{f_{q|\boldsymbol{Z}_{-1,l}}(0|\boldsymbol{Z}_{l,i}^{(j,k)})}, \boldsymbol{Z}_{l,i}^{(j,k)}\right) \in \cdot\right\}$$

is a Poisson process with the mean measure

$$\mu_{3,l}^{(j,k)}(dq, d\boldsymbol{z}) = \mu_{2,l}^{(j,k)} \circ \mathcal{T}_l^{-1}(dq, d\boldsymbol{z}) = f_{q_l|\boldsymbol{Z}_{-1,l}}(0|\boldsymbol{z})dq \cdot f_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z})d\boldsymbol{z} \tag{B.54}$$

on $\mathbb{R}_{s_l^{(j)}} \times \mathcal{Z}_{-1,l}$. Finally, let $F_l^{(j,k)}(\cdot|\boldsymbol{z})$ be the conditional distribution function of $\xi^{(j,k)}$ given $q_l = 0$ and $\boldsymbol{Z}_{-1,l} = \boldsymbol{z}$, which makes its density function be $f_{\xi^{(i,j)}|(q_l,\boldsymbol{Z}_{-1,l})}(\xi \mid 0, \boldsymbol{z})$. Let $\{\xi_i^{(j,k)}\}_{i=1}^\infty$ be an i.i.d. sequence follows the conditional distribution $F_l^{(j,k)}(\cdot|\boldsymbol{Z}_{l,i}^{(j,k)})$. Then by

applying again Proposition 3.7 in Resnick (2008), the composed point process

$$\mathbf{N}_l^{(j,k)}(\cdot) = \sum_{i=1}^{\infty} \mathbb{1}\left\{ \left(\frac{\mathcal{J}_{l,i}^{(j,k)}}{f_{q|\boldsymbol{Z}_{-1,l}}(0|\boldsymbol{Z}_{l,i}^{(j,k)})}, \boldsymbol{Z}_{l,i}^{(j,k)}, \xi_i^{(j,k)} \right) \in \cdot \right\} \tag{B.55}$$

is a Poisson process with the mean measure

$$\begin{aligned}\mu_l^{(j,k)}(dq, d\boldsymbol{z}, d\xi) &= \mu_{3,l}^{(j,k)}(dq, d\boldsymbol{z}) F_l^{(j,k)}(d\xi|\boldsymbol{z}) \\ &= f_{\xi^{(i,j)}|(q_l, \boldsymbol{Z}_{-1,l})}(\xi \mid 0, \boldsymbol{z})\, f_{q_l|\boldsymbol{Z}_{-1,l}}(0 \mid \boldsymbol{z})\, f_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z}) \mathrm{d}q \mathrm{d}\boldsymbol{z} \mathrm{d}\xi,\end{aligned}$$

which matches the desired mean measure (B.50).

In summary, through Steps (I)-(III) we derive that for each $l \in \{1,2\}$ and $(j,k) \in \mathcal{S}(l)$, it holds that $\widehat{\mathbf{N}}_{l,T}^{(j,k)} \Rightarrow \mathbf{N}_l^{(j,k)}$ in $M_p(E_l)$ as $T \to \infty$, where $\mathbf{N}_l^{(j,k)}$ is a Poisson point process with the representation (B.55).

Part 3: Asymptotical independence of point processes.

We now show that the empirical point processes $\left\{ \widehat{\mathbf{N}}_{l,T}^{(j,k)}, l \in \{1,2\}, (j,k) \in \mathcal{S}(l) \right\}$ are asymptotically independent, that is, for any compact sets $\{F_l^{(j,k)} \in \mathcal{E}_l, l \in \{1,2\}, (j,k) \in \mathcal{S}(l)\}$ and non-negative integers $\{k_l^{(j,k)}, l \in \{1,2\}, (j,k) \in \mathcal{S}(l)\}$, it holds that

$$\begin{aligned}&\mathbb{P}\left\{ \bigcap_{(l,j,k) \in \mathcal{I}_s} \left(\widehat{\mathbf{N}}_{l,T}^{(j,k)}(F_l^{(j,k)}) = k_l^{(j,k)} \right) \right\} \\ &\to \prod_{(l,j,k) \in \mathcal{I}_s} \frac{\exp\left(-\mu_l^{(j,k)}(F_l^{(j,k)})\right) \left\{ \mu_l^{(j,k)}(F_l^{(j,k)}) \right\}^{k_l^{(j,k)}}}{k_l^{(j,k)}!},\end{aligned} \tag{B.56}$$

as $T \to \infty$, where $\mathcal{I}_s$ is any subset of $\mathcal{I} = \{(l,j,k) : l \in \{1,2\}, (j,k) \in \mathcal{S}(l)\}$.

Suppose that $|\mathcal{I}_s| = n, 1 \le n \le |\mathcal{I}|$. For notational simplicity, we label the n triples $\left\{ \left(\widehat{\mathbf{N}}_{l,T}^{(j,k)}, F_l^{(j,k)}, k_l^{(j,k)} \right), (l,j,k) \in \mathcal{I}_s \right\}$ as $\left\{ (\widehat{\mathbf{N}}_{i,T}, F_i, k_i), 1 \le i \le n \right\}$, and define

$$\widehat{\mathbf{C}}_T = \sum_{i=1}^{n} \widehat{\mathbf{N}}_{i,T}(F_i) = \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{1}\left\{ (Tq_{i,t}, \boldsymbol{Z}_{-1,i,t}, \xi_i) \in F_i \right\} =: \sum_{t=1}^{T} \widehat{\mathbf{C}}_t, \quad \text{say.} \tag{B.57}$$

Let $A_{i,t}^T$ be the event $\{(Tq_{i,t}, \boldsymbol{Z}_{-1,i,t}, \xi_i) \in F_{i,t}\}$ and $B_t^T = \bigcup_{i=1}^n A_{i,t}^T$, namely B_t^T occurs if and only if at least one of $\{A_{i,t}\}_{i=1}^n$ occurs. The derivation for (B.56) includes two steps. First, we calculate $\lim_{T\to\infty} \mathbb{P}(\widehat{\mathbf{C}}_T = k)$, for which we show $\mathbb{P}(A_{i,t}^T \cap A_{i',t}^T) = O(T^{-2})$ as $T \to \infty$. In the second step, we calculate $\lim_{T\to\infty} \mathbb{P}\left\{ \bigcap_{i=1}^n \left(\widehat{\mathbf{N}}_{i,T}(F_i) = k_i \right) \mid \widehat{\mathbf{C}}_T = k \right\}$ with $\sum_{i=1}^n k_i = k$, using the arguments of thinning and blocking.

Step 1. In this step, we first show that for each $1 \le t \le T$, the distinct events $A_{i,t}^T$ and $A_{i',t}^T$ cannot happen together asymptotically. Suppose that

$$\begin{aligned}A_{i,t}^T &= \left\{ (Tq_{l,t}, \boldsymbol{Z}_{-1,l,t}, \xi_t^{(j,k)}) \in F_i = F_{1,i} \times F_{2,i} \times F_{3,i} \right\} \text{ and} \\ A_{i',t}^T &= \left\{ (Tq_{l',t}, \boldsymbol{Z}_{-1,l',t}, \xi_t^{(j',k')}) \in F_{i'} = F_{1,i'} \times F_{2,i'} \times F_{3,i'} \right\},\end{aligned} \tag{B.58}$$

respectively. First, consider the case that $l = l'$ and $(j,k) \neq (j',k')$. We notice that since both (j,k) and (j',k') belong to $\mathcal{S}(l)$, then either (i) $j = k'$ and $j' = k$ or (ii) $\{j,k\} \cap \{j',k'\} = \emptyset$. Under (i) we have $\mathbb{P}(Tq_{l,t} \in F_{1,i} \cap F_{1,i'}) = 0$, since $F_{1,i} \subset \mathbb{R}_{s_l^j}$ and $F_{1,i'} \subset \mathbb{R}_{s_l^k}$, while $s_l^j = -s_l^k$. Also, since $\xi_t^{(j,k)}\xi_t^{(j',k')} = 0$ under (ii), $\mathbb{P}(\xi_t^{(j,k)} \in F_{3,i}, \xi_t^{(j',k')} \in F_{3,i'}) = 0$. In summary, $\mathbb{P}(A_{i,t}^T \cap A_{i',t}^T) = 0$ if $l = l'$ and $(j,k) \neq (j',k')$.

On the other hand, if $l \neq l'$

$$\begin{aligned}
\mathbb{P}(A_{i,t}^T \cap A_{i',t}^T) =& \mathbb{P}\left(\left\{(Tq_{l,t}, \boldsymbol{Z}_{-1,l,t}, \xi_t^{(j,k)}) \in F_i\right\} \cap \left\{(Tq_{l',t}, \boldsymbol{Z}_{-1,l',t}, \xi_t^{(j',k)}) \in F_{i'}\right\}\right) \\
\leq& \mathbb{P}\left(\{Tq_{l,t} \in F_{1,i}\} \cap \{Tq_{l',t} \in F_{1,i'}\}\right) \\
=& \mathbb{E}_{\boldsymbol{Z}_{-1,l},\boldsymbol{Z}_{-1,l'}}\left\{\int_{Tq \in F_{1,i}, Tq' \in F_{1,i'}} f_{(q_l,q_{l'})|(\boldsymbol{Z}_{-1,l},\boldsymbol{Z}_{-1,l'})}(q,q')dqdq'\right\} \\
=& \frac{1}{T^2}\mathbb{E}_{\boldsymbol{Z}_{-1,l},\boldsymbol{Z}_{-1,l'}}\left\{\int_{\tilde{q} \in F_{1,i}, \tilde{q}' \in F_{1,i'}} f_{(q_l,q_{l'})|(\boldsymbol{Z}_{-1,l},\boldsymbol{Z}_{-1,l'})}\left(\frac{\tilde{q}}{T}, \frac{\tilde{q}'}{T}\right) d\tilde{q}d\tilde{q}'\right\} \\
=& O(T^{-2}) \quad \text{as } T \to \infty. \qquad \text{(B.59)}
\end{aligned}$$

Therefore, we obtain that $\mathbb{P}(A_{i,t}^T \cap A_{i',t}^T) = O(T^{-2})$ as $T \to \infty$ if $i \neq i'$.

Note that by the inclusion-exclusion principle,

$$\begin{aligned}
\mathbb{P}(B_t^T) =& \mathbb{P}\left(\bigcup_{i=1}^n A_{i,t}^T\right) \\
=& \sum_{i=1}^n \mathbb{P}(A_{i,t}^T) + \sum_{k=2}^n (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbb{P}(A_{i_1}^T \cap \cdots \cap A_{i_k}^T). \qquad \text{(B.60)}
\end{aligned}$$

Because $\mathbb{P}(A_{i_1}^T \cap \cdots \cap A_{i_k}^T) \leq \mathbb{P}(A_{i_1}^T \cap A_{i_2}^T)$, from (B.59) and (B.60) it yields that

$$\mathbb{P}(B_t^T) = \sum_{i=1}^n \mathbb{P}(A_{i,t}^T) + O(T^{-2}). \qquad \text{(B.61)}$$

From (B.49) we have

$$\mathbb{P}(A_{i,t}^T) = \mu_i(F_i)/T + o(T^{-1}), \qquad \text{(B.62)}$$

which implies that

$$\mathbb{P}(B_t^T) = \sum_{i=1}^n \mu_i(F_i)/T + o(T^{-1}). \qquad \text{(B.63)}$$

With the similar arguments used in Step 2 of Part 2, we can verify the conditions for Meyer's theorem for $\{B_t^T\}_{t=1}^T$, which delivers that for any $0 \leq k \leq T$,

$$\mathbb{P}\left\{\text{exactly } k \text{ of } \{B_t^T\}_{t=1}^T \text{ occur}\right\} \to \frac{\exp\left(-\sum_{i=1}^n \mu_i(F_i)\right)\{\sum_{i=1}^n \mu_i(F_i)\}^k}{k!},$$

as $T \to \infty$. We notice that

$$\left\{\widehat{\mathbf{C}}_T = k\right\} / \left\{\text{exactly } k \text{ of } \{B_t^T\}_{t=1}^T \text{ occur}\right\} \subset \left\{\text{for some } 1 \leq t \leq T, \widehat{\mathbf{C}}_t \geq 2\right\}$$

and

$$\sum_{t=1}^T \mathbb{P}(\widehat{\mathbf{C}}_t \geq 2) \leq \sum_{t=1}^T \sum_{1 \leq i \neq i' \leq n} \mathbb{P}(A_{i_1}^T \cap A_{i_2}^T) = O(nT^{-1}),$$

where n is finite, since it is the cardinality of $\mathcal{I}_s$. Hence, we obtain

$$\mathbb{P}(\widehat{\mathbf{C}}_T = k) = \mathbb{P}\left\{\text{exactly } k \text{ of } \{B_t^T\}_{t=1}^T \text{ occur}\right\} + o(1)$$
$$\to \frac{\exp\left(-\sum_{i=1}^n \mu_i(F_i)\right)\{\sum_{i=1}^n \mu_i(F_i)\}^k}{k!}, \quad \text{as } T \to \infty. \tag{B.64}$$

Step 2. Now we turn to calculate $\mathbb{P}\left\{\bigcap_{i=1}^n \left(\widehat{\mathbf{N}}_{i,T}(F_i) = k_i\right)\right\}$. Let $k = \sum_{i=1}^n k_i$. Note that

$$\mathbb{P}\left\{\bigcap_{i=1}^n \left(\widehat{\mathbf{N}}_{i,T}(F_i) = k_i\right)\right\}$$
$$= \mathbb{P}\left(\widehat{\mathbf{C}}_T = k\right)\mathbb{P}\left\{\bigcap_{i=1}^n \left(\widehat{\mathbf{N}}_{i,T}(F_i) = k_i\right) \mid \widehat{\mathbf{C}}_T = k\right\}$$
$$= \mathbb{P}\left(\widehat{\mathbf{C}}_T = k\right)\left[\mathbb{P}\left\{\bigcap_{i=1}^n \left(k_i \text{ of } \{A_{i,t}^T\}_{t=1}^T \text{ are assigned}\right) \mid k \text{ of } \{B_t^T\}_{t=1}^T \text{ occur}\right\} + o(1)\right]$$
$$=: P_{1,T} \times P_{2,T} + o(1), \text{ say.}$$

For $P_{1,T}$, by (B.64) we have

$$P_{1,T} \to \frac{\exp\left(-\sum_{i=1}^n \mu_i(F_i)\right)\{\sum_{i=1}^n \mu_i(F_i)\}^{\sum_{i=1}^n k_i}}{(\sum_{i=1}^n k_i)!}, \tag{B.65}$$

as $T \to \infty$. We now proceed to obtain the limits of $P_{2,T}$ by the blocking argument as in Meyer (1973).

Specifically, for any positive integer m, partition the observation indices into consecutive blocks of p_m and q_m alternately, where p_m and q_m are the same as those in Step (2) of Part 2, beginning with the initial block $\{1, \cdots, p_m\}$. Let P_m and Q_m denote those indices falling into size p_m and q_m blocks, respectively, and $t_m = m(p_m + q_m)$. Let $I_t^{i,t_m} = \left\{A_{i,t}^{t_m} \text{ happens if } B_t^{t_m} \text{ happens}\right\}$. According to (B.62) and (B.63),

$$\mathbb{P}(I_t^{i,t_m}) = \mathbb{P}(A_{i,t}^{t_m} | B_t^{t_m}) = \frac{\mathbb{P}(A_{i,t}^{t_m} \cap B_t^{t_m})}{\mathbb{P}(B_t^{t_m})} = \frac{\mathbb{P}(A_{i,t}^{t_m})}{\mathbb{P}(B_t^{t_m})}$$
$$= \frac{\mu_i(F_i)}{\sum_{i=1}^n \mu_i(F_i)} + o(1) =: p_i + o(1), \text{ say,}$$

as $m \to \infty$.

Let $\mathcal{G}_k = \{G_k = \{j_s\}_{s=1}^k : 1 \le j_1 \le \cdots \le j_k \le t_m\}$ be the collection of the subsets of $\{1, \cdots, t_m\}$ with the cardinality k. Then,

$$\left\{k \text{ of } \{B_t^{t_m}\}_{t=1}^T \text{ occur}\right\} = \cup_{G_k \in \mathcal{G}_k}\{B_t^{t_m} \text{ occur iff } t \in G_k\}, \tag{B.66}$$

where "iff" is short for "if and only if". For each $G_k = \{j_s\}_{s=1}^k \in \mathcal{G}_k$, let

$$H(G_k) = \{(H_i = \{j_{i_s}\}_{s=1}^{k_i})_{i=1}^n : \cup_{i=1}^n H_i = G_k \text{ and } H_i \cap H_{i'} = \emptyset \text{ if } i \ne i'\}$$

be the collection of all possible n-partitions of G_k with each segment H_i containing k_i indices of G_k. Then we note that $|H(G_k)| = k!/(\prod_{i=1}^n k_i!)$.

$$\mathbb{P}\{\cap_{i=1}^n (k_i \text{ of } \{A_{i,t}^{t_m}\}_{t=1}^{t_m} \text{ are assigned}) \mid B_t^{t_m} \text{ occur iff } t \in G_k\}$$

$$= \sum_{(H_i)_{i=1}^n \in H(G_k)} \mathbb{P}(\cap_{i=1}^n \cap_{s=1}^{k_i} I_{j_{i_s}}^{i,t_m}). \tag{B.67}$$

By inspecting the proof of Theorem 1 of Meyer (1973), we find that if k of $\{B_t^{t_m}\}_{t=1}^{t_m}$ happens, then asymptotically all the k indices lie in separate blocks in P_m, implying that any $|j-j'| \geq q_m$ for any $j \neq j' \in G_k$. Therefore, for large enough m, we have

$$\left| \mathbb{P}(\cap_{i=1}^n \cap_{s=1}^{k_i} I_{j_{i_s}}^{i,t_m}) - \prod_{i=1}^n \prod_{s=1}^{k_i} \mathbb{P}(I_{j_{i_s}}^{i,t_m}) \right| \leq k\alpha_{t_m}(q_m),$$

by applying the definition of the α-mixing coefficients repeatedly for k times. Since $|H(G_k)| = k!/(\prod_{i=1}^n k_i!)$ and $\mathbb{P}(I_{j_{i_s}}^{i,t_m}) = p_i + o(1)$, we obtain

$$\left| \sum_{\mathcal{C}_*} \mathbb{P}\Big(\bigcap_{i\in[n],t\in[k_i]} I_{j_t}^{i,t_m} \Big) - \frac{k!}{\prod_{i=1}^n k_i!} \prod_{i=1}^n p_i^{k_i} + o(1) \right| \leq k \frac{k!}{\prod_{i=1}^n k_i!} \alpha_{t_m}(q_m) = o(1), \tag{B.68}$$

where the last equality is due to that k is a given integer and $\alpha_{t_m}(q_m) \to 0$ as $m \to \infty$. Combining (B.67) and (B.68) leads to

$$\mathbb{P}\{\cap_{i=1}^n (k_i \text{ of } \{A_{i,t}^{t_m}\}_{t=1}^{t_m} \text{ are assigned}) \mid B_t^{t_m} \text{ occur iff } t \in G_k\} = \frac{k!}{\prod_{i=1}^n k_i!} \prod_{i=1}^n p_i^{k_i} + o(1),$$

for each $G_k \in \mathcal{G}_k$. This together with (B.66) yields that

$$P_{2,t_m} = \frac{k!}{\prod_{i=1}^n k_i!} \prod_{i=1}^n p_i^{k_i} + o(1), \quad \text{as } m \to \infty,$$

where $P_{2,t_m} = \mathbb{P}\{\cap_{i=1}^n (k_i \text{ of } \{A_{i,t}^{t_m}\}_{t=1}^{t_m} \text{ are assigned}) \mid k \text{ of } \{B_t^{t_m}\}_{t=1}^{t_m} \text{ occur}\}$. Since for any T, there exists a m such that $T \in [t_m, t_{m+1})$, the above result implies that

$$P_{2,T} \to \frac{k!}{\prod_{i=1}^n k_i!} \prod_{i=1}^n p_i^{k_i} + o(1) = \frac{(\sum_{i=1}^n k_i)!}{\prod_{i=1}^n k_i!} \prod_{i=1}^n \left\{ \frac{\mu_i(F_i)}{\sum_{i=1}^n \mu_i(F_i)} \right\}^{k_i}, \quad \text{as } T \to \infty, \tag{B.69}$$

since $k = \sum_{i=1}^n k_i$ and $p_i = \mu_i(F_i)/(\sum_{i=1}^n \mu_i(F_i))$. Combining (B.65) with (B.69) yields that

$$\mathbb{P}\left\{ \bigcap_{i=1}^n \left(\widehat{\mathbf{N}}_{i,T}(F_i) = k_i \right) \right\} = P_{1,T} P_{2,T} + o(1) = \prod_{i=1}^n \frac{\exp\left(-\mu_i(F_i)\right) \{\mu_i(F_i)\}^{k_i}}{k_i!} + o(1),$$

which proves (B.56) and implies that $\left(\widehat{\mathbf{N}}_{l,T}^{(j,k)}, l \in \{1,2\}, (j,k) \in \mathcal{S}(l) \right)$ are asymptotically independent. These together with Part 2 conclude that $\widehat{\mathbf{N}}_{l,T}^{(j,k)} \Rightarrow \mathbf{N}_l^{(j,k)}$ in $M_p(E_l)$ as $T \to \infty$, where $\left(\mathbf{N}_l^{(j,k)}, l \in \{1,2\}, (j,k) \in \mathcal{S}(l) \right)$ are independent Poisson point processes with the representation (B.55).

Part 4: Continuous mapping.

In this part, we show that $\mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)}\left(\widehat{\mathbf{N}}_{l,T}^{(j,k)} \right) \xrightarrow{d} \mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)}\left(\mathbf{N}_l^{(j,k)} \right)$ as $T \to \infty$. If $\mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)}(\cdot)$ is a continuous functional in $M_p(E_l)$, then it follows by the continuous mapping thoerem. To show that $\mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)}(\cdot)$ is continuous mapping from $M_p(E_l)$ to $\mathbb{R}$, we use Proposition 3.13 in

Resnick (2008), which requires $\mathcal{T}^{(j,k)}_{l,\boldsymbol{v}_l}(\cdot)$ has a compact support. Therefore, we use a truncation argument. Recall that for any $\mathbf{N} \in M_p(E)$,

$$\mathcal{T}^{(j,k)}_{l,\boldsymbol{v}_l}(\mathbf{N}) = \int_{E_l} g^{(j,k)}_{l,\boldsymbol{v}_l}(x,\boldsymbol{y},z) d\mathbf{N}(x,\boldsymbol{y},z),$$

where $x \in \mathbb{R}_{s_l^{(j)}}, \boldsymbol{y} \in \mathcal{Z}_{-1,l}, z \in \mathbb{R}$, and

$$g^{(j,k)}_{l,\boldsymbol{v}_l}(x,\boldsymbol{y},z) = z \cdot \mathbb{1}\left\{ s_l^{(j)}(x + \boldsymbol{y}^{\mathrm{T}}\boldsymbol{v}_{-1,l}) \le 0 < s_l^{(j)} x \right\}.$$

Therefore, the support of $\mathcal{T}^{(j,k)}_{l,\boldsymbol{v}_l}$ is $\mathcal{Q}_l^{(j)} \times \mathcal{Z}_{-1,l} \times \mathbb{R}$, where $\mathcal{Q}_l^{(j)} = \{q : s_l^{(j)}(q + \boldsymbol{y}^{\mathrm{T}}\boldsymbol{v}_{-1,l}) \le 0 < s_l^{(j)} q \text{ for some } \boldsymbol{y} \in \mathcal{Z}_{-1,l}\}$, which is compact since $\mathcal{Z}_{-1,l}$ is compact. For any $M > 0$, we let $E_{l,M} = \left\{(x,\boldsymbol{y},z) : x \in \mathbb{R}_{s_l^{(j)}}, \boldsymbol{y} \in \mathcal{Z}_{-1,l}, |z| < M\right\}$, which is a compact set. Let

$$R_T = \mathcal{T}^{(j,k)}_{l,\boldsymbol{v}_l}\left(\widehat{\mathbf{N}}^{(j,k)}_{l,T}\right) = \int_{E_l} g^{(j,k)}_l(x,\boldsymbol{y},z) d\widehat{\mathbf{N}}^{(j,k)}_{l,T}(x,\boldsymbol{y},z),$$

$$R_{T,M} = \int_{E_{l,M}} g^{(j,k)}_l(x,\boldsymbol{y},z) d\widehat{\mathbf{N}}^{(j,k)}_{l,T}(x,\boldsymbol{y},z),$$

$$R_{0,M} = \int_{E_{l,M}} g^{(j,k)}_l(x,\boldsymbol{y},z) d\mathbf{N}^{(j,k)}_{l,T}(x,\boldsymbol{y},z) \text{ and}$$

$$R_0 = \mathcal{T}^{(j,k)}_{l,\boldsymbol{v}_l}\left(\mathbf{N}^{(j,k)}_{l,T}\right) = \int_{E_l} g^{(j,k)}_l(x,\boldsymbol{y},z) d\mathbf{N}^{(j,k)}_{l,T}(x,\boldsymbol{y},z).$$

In the following, we show in three steps that (i) $R_{T,M} \xrightarrow{d} R_{0,M}$ for any fixed $M > 0$ as $T \to \infty$ by the continuous mapping theorem, (ii) $\lim_{M\to\infty} \limsup_{T\to\infty} \mathbb{P}\{|R_T - R_{T,M}| > \varepsilon\} \to 0$ for any $\varepsilon > 0$, and (iii) $R_{0,M} \xrightarrow{d} R_0$ as $M \to \infty$. Then by Theorem 4.2 of Billingsley (1968), $R_T \xrightarrow{d} R_0$ as $T \to \infty$.

Step (1). For any fixed $M > 0$, let $\mathcal{M}^{(j,k)}_{l,\boldsymbol{v}_l}(\mathbf{N}) = \int_{E_{l,M}} g^{(j,k)}_{l,\boldsymbol{v}_l}(x,\boldsymbol{y},z) d\mathbf{N}(x,\boldsymbol{y},z)$ for any $\mathbf{N} \in M_p(E)$. By Proposition 3.13 in Resnick (2008), if any sequence $\mathbf{N}_n \Rightarrow \mathbf{N}$, then the points of $\mathbf{N}_n$ locating in $E_{l,m}$ converge to that of $\mathbf{N}$ locating in $E_{l,m}$. Since restricted on $E_{l,M}$, the function $g^{(j,k)}_l(x,\boldsymbol{y},z)$ has a compact support but is discontinuous at $x = 0$ or $x + \boldsymbol{y}^{\mathrm{T}}\boldsymbol{v}_{-1,l} = 0$, the functional $\mathcal{M}^{(j,k)}_{l,\boldsymbol{v}_l}$ is continuous except on

$$\mathcal{D}(\mathcal{M}^{(j,k)}_{l,\boldsymbol{v}_l}) = \{\mathbf{N} \in M_p(E) : x_i^{\mathbf{N}} = 0 \text{ or } x_i^{\mathbf{N}} + (\boldsymbol{y}_i^{\mathbf{N}})^{\mathrm{T}}\boldsymbol{v}_{-1,l} = 0 \text{ for some } i \ge 1\},$$

where $(x_i^{\mathbf{N}}, \boldsymbol{y}_i^{\mathbf{N}}, z_i^{\mathbf{N}}, i \ge 1)$ denote the points of $\mathbf{N}$. Since

$$\mathbb{P}\left\{\mathbf{N}^{(j,k)}_l \in \mathcal{D}(\mathcal{M}^{(j,k)}_{l,\boldsymbol{v}_l})\right\} = \mathbb{P}\left\{\exists\, i, J^{(j,k)}_{l,i} = 0 \text{ or } J^{(j,k)}_{l,i} + (\boldsymbol{Z}^{(j,k)}_{l,i})^{\mathrm{T}}\boldsymbol{v}_{-1,l} = 0\right\} = 0 \quad \text{(B.70)}$$

and $J^{(j,k)}_{l,i}$ is absolutely continuous, we have

$$R_{T,M} = \mathcal{M}^{(j,k)}_{l,\boldsymbol{v}_l}\left(\widehat{\mathbf{N}}^{(j,k)}_{l,T}\right) \xrightarrow{d} \mathcal{M}^{(j,k)}_{l,\boldsymbol{v}_l}\left(\mathbf{N}^{(j,k)}_l\right) = R_{0,M}, \quad \text{(B.71)}$$

for any fixed $M > 0$ as $T \to \infty$, by the continuous mapping theorem.

Step (2). Next, we show that

$$\lim_{M\to\infty} \limsup_{T\to\infty} \mathbb{P}\{|R_T - R_{T,M}| > \varepsilon\} \to 0, \quad \text{(B.72)}$$

for any $\varepsilon > 0$. For notational simplicity, we denote $\xi_t = \xi_t^{(j,k)}$, $q_t = q_{l,t}$, $\boldsymbol{Z}_{-1,t} = \boldsymbol{Z}_{-1,l,t}$, and suppose $s_l^{(j)} = 1$ without loss of generality. Then, for any $M > 0$,

$$\begin{aligned}|R_T - R_{T,M}| \leq & \sum_{t=1}^{T} \left\{ |\xi_t| \mathbb{1}\left(|\xi_t| \geq M\right) \mathbb{1}\left(Tq_t + \boldsymbol{Z}_{-1,t}^{\mathrm{T}} \boldsymbol{v}_{-1,l} \leq 0 < Tq_t\right) \right\} \\ =: & \sum_{t=1}^{T} G_t(M), \quad \text{say.} \end{aligned} \tag{B.73}$$

Since

$$\begin{aligned}\mathbb{E}\left\{ |\xi_t| \mathbb{1}\left(|\xi_t| \geq M\right) \mid \boldsymbol{Z}_{l,t}^{\mathrm{T}} \boldsymbol{\gamma} = 0 \right\} \leq & \{\mathbb{E}(|\xi_t|^2 | \boldsymbol{Z}_{l,t}^{\mathrm{T}} \boldsymbol{\gamma} = 0)\}^{1/2} \left\{ \mathbb{P}(|\xi_t| > M | \boldsymbol{Z}_{l,t}^{\mathrm{T}} \boldsymbol{\gamma} = 0) \right\}^{1/2} \\ \leq & \{\mathbb{E}(|\xi_t|^2 | \boldsymbol{Z}_{l,t}^{\mathrm{T}} \boldsymbol{\gamma} = 0)\}^{1/2} \frac{\{\mathbb{E}(|\xi_t|^2 | \boldsymbol{Z}_{l,t}^{\mathrm{T}} \boldsymbol{\gamma} = 0)\}^{1/2}}{M} \\ = & O_p\left(M^{-1}\right) \end{aligned} \tag{B.74}$$

almost surely, where the first inequality is via Cauchy-Schwarz inequality and the second is by Markov inequality, provided $\mathbb{E}(|\xi_t|^2 | \boldsymbol{Z}_{l,t}^{\mathrm{T}} \boldsymbol{\gamma} = 0) < \infty$ for $\boldsymbol{\gamma}$ in a neighborhood of $\boldsymbol{\gamma}_{l,0}$, which is ensured by Assumption 4 (iv). Using (B.74) and with the similar arguments as in the proof of Lemma A.2 (i), we can show that $\mathbb{E}\left\{G_t(M)\right\} = O\left((MT)^{-1}\right)$. Therefore,

$$\mathbb{E}|R_T - R_{T,M}| \leq \sum_{t=1}^{T} \mathbb{E}\left\{G_t(M)\right\} = O\left(M^{-1}\right),$$

for any T and M, which implies (B.72) by Markov inequality.

Step (3). Next, we show that $R_0 = R_{0,M} + o_p(1)$. We notice that

$$R_0 - R_{0,M} = \sum_{i=1}^{\infty} \left[\xi_i^{(j,k)} \mathbb{1}\left(|\xi_i^{(j,k)}| > M\right) \mathbb{1}\left\{ J_{l,i}^{(j,k)} + \left(\boldsymbol{Z}_{l,i}^{(j,k)}\right)^{\mathrm{T}} \boldsymbol{v}_{-1,l} \leq 0 < J_{l,i}^{(j,k)} \right\} \right].$$

Let $Z_{\max} = \max\left\{ -\left(\boldsymbol{Z}_{l,i}^{(j,k)}\right)^{\mathrm{T}} \boldsymbol{v}_{-1,l}, \boldsymbol{Z}_{l,i}^{(j,k)} \in \mathcal{Z}_{-1,l} \right\}$, which is bounded since both $\mathcal{Z}_{-1,l}$ and the space of $\boldsymbol{v}_{-1,l}$ are compact. This means that $Z_{\max} < \infty$. Since $f_{q|\boldsymbol{Z}_{-1,l}}(0|\boldsymbol{Z}_{-1,l})$ is uniformly bounded by some constant, say F_l, by Assumption 5 (ii), the event

$$\frac{\mathcal{J}_{l,i}^{(j,k)}}{f_{q|\boldsymbol{Z}_{-1,l}}(0|\boldsymbol{Z}_{l,i}^{(j,k)})} + \left(\boldsymbol{Z}_{l,i}^{(j,k)}\right)^{\mathrm{T}} \boldsymbol{v}_{-1,l} \leq 0 < \frac{\mathcal{J}_{l,i}^{(j,k)}}{f_{q|\boldsymbol{Z}_{-1,l}}(0|\boldsymbol{Z}_{l,i}^{(j,k)})} \equiv J_{l,i}^{(j,k)}$$

implies $0 \leq \mathcal{J}_{l,i}^{(j,k)} \leq F_l Z_{\max}$. Therefore,

$$|R_0 - R_{0,M}| \leq \sum_{i=1}^{\infty} \left\{ \xi_i^{(j,k)} \mathbb{1}\left(|\xi_i^{(j,k)}| > M\right) \mathbb{1}(0 \leq \mathcal{J}_{l,i}^{(j,k)} \leq F_l Z_{\max}) \right\}. \tag{B.75}$$

Note that $\mathcal{J}_{l,i}^{(j,k)} = \sum_{n=1}^{i} \mathcal{E}_{l,n}^{(j,k)}$ where $\{\mathcal{E}_{l,n}^{(j,k)}\}_{n=1}^{\infty}$ is an i.i.d. sequence of unit-exponential variables, and $\{\xi_i^{(j,k)}\}_{i=1}^{\infty}$ be an i.i.d. sequence follows the conditional distribution $F_l^{(j,k)}(\cdot|\boldsymbol{Z}_{l,i}^{(j,k)})$, that is independent to $\{\mathcal{E}_{l,n}^{(j,k)}\}_{n=1}^{\infty}$. Hence, $\mathcal{P}(t) = \sum_{i=1}^{N(t)} \xi_i^{(j,k)} \mathbb{1}(|\xi_i^{(j,k)}| > M)$ for $t \geq 0$

is compound Poisson process with the jump size $\xi_i^{(j,k)}\mathbb{1}(|\xi_i^{(j,k)}| > M)$, where $N(t) = \sum_{i=1}^{\infty}\mathbb{1}(\mathcal{J}_{l,i}^{(j,k)} \le t)$ is a homogeneous Poisson process with rate 1. Therefore, we have

$$\begin{aligned}\mathbb{E}\left\{|R_0 - R_{0,M}|\right\} &\overset{(i)}{\le} F_l Z_{\max}\mathbb{E}\left\{\xi_i^{(j,k)}\mathbb{1}\left(|\xi_i^{(j,k)}| > M\right)\right\},\\ &\overset{(ii)}{\le} F_l Z_{\max}\sqrt{\mathbb{E}\left\{\left(\xi_i^{(j,k)}\right)^2\right\}}\sqrt{\mathbb{P}\left(|\xi_i^{(j,k)}| > M\right)}\\ &\overset{(iii)}{\le} F_l Z_{\max}\mathbb{E}\left\{\left(\xi_i^{(j,k)}\right)^2\right\}/M \to 0, \quad \text{as } M \to \infty, \end{aligned} \tag{B.76}$$

where (i) is from Wald's identity (Wald, 1944), (ii) is from Cauchy-Schwartz's inequality and (iii) is from Markov's inequality. Because of the above result, we obtain $R_{0,M} = R_0 + o_p(1)$, which further implies that $R_{0,M} \xrightarrow{d} R_0$.

Through the three steps we have shown that (i) $R_{T,M} \xrightarrow{d} R_{0,M}$ for any fixed $M > 0$ as $T \to \infty$ by the continuous mapping theorem, (ii) $\lim_{M\to\infty}\limsup_{T\to\infty}\mathbb{P}\left\{|R_T - R_{T,M}| > \varepsilon\right\} \to 0$ for any $\varepsilon > 0$, and (iii) $R_{0,M} \xrightarrow{d} R_0$ as $M \to \infty$. Therefore, by applying Theorem 4.2 of Billingsley (1968), $R_T \xrightarrow{d} R_0$ as $T \to \infty$, i.e., $\mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)}\left(\widehat{\mathbf{N}}_{l,T}^{(j,k)}\right) \xrightarrow{d} \mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)}\left(\mathbf{N}_{l,T}^{(j,k)}\right)$. Because in Part 3 it is shown that $\left(\widehat{\mathbf{N}}_{l,T}^{(j,k)}, l \in \{1,2\}, (j,k) \in \mathcal{S}(l)\right)$ are asymptotically independent, we conclude that

$$\sum_{l=1}^{L}\sum_{(j,k)\in\mathcal{S}(l)}\mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)}\left(\widehat{\mathbf{N}}_{l,T}^{(j,k)}\right) \xrightarrow{d} \sum_{l=1}^{L}\sum_{(j,k)\in\mathcal{S}(l)}\mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)}\left(\mathbf{N}_{l,T}^{(j,k)}\right), \tag{B.77}$$

as $T \to \infty$, which concludes the proof. □

B.8. Proof of Lemma B.3.

PROOF. The proof for this lemma adapts that in Chernozhukov and Hong (2004). First, we decompose $D_T(\boldsymbol{v}) = \sum_{l=1}^{2}\sum_{(j,k)\in\mathcal{S}(l)} D_T^{(j,k)}(\boldsymbol{v}_l)$, where $\boldsymbol{v}_l \in \mathbb{R}^{d_l}$ for $l = 1, 2$, and

$$D_T^{(j,k)}(\boldsymbol{v}_l) = \sum_{t=1}^{T}\xi_t^{(j,k)}\mathbb{1}\left\{s_l^{(j)}\left(Tq_{l,t} + \boldsymbol{Z}_{-1,l,t}^{\mathrm{T}}\boldsymbol{v}_{-1,l}\right) \le 0 < s_l^{(j)}Tq_{l,t}\right\}.$$

It is sufficient to show that $D_T^{(j,k)}(\boldsymbol{v}_l)$ is stochastic equi-lower-semicontinuous for each $l \in \{1,2\}$ and $(j,k) \in \mathcal{S}(l)$. Without loss of generality, we take $l = 1, j = 1, k = 2$, since the other cases can be proved in the same way. To simplify notations, let $\tilde{\boldsymbol{v}} = \boldsymbol{v}_1, \tilde{q}_t = q_{1,t}, \tilde{\boldsymbol{Z}}_t = \boldsymbol{Z}_{-1,l,t}, \tilde{\xi}_t = \xi_t^{(j,k)}$ and $\tilde{D}_T(\tilde{\boldsymbol{v}}) = D_T^{(1,2)}(\boldsymbol{v}_1)$. With the above notations,

$$\tilde{D}_T(\tilde{\boldsymbol{v}}) = \sum_{t=1}^{T}\tilde{\xi}_t\mathbb{1}\left\{\left(T\tilde{q}_t + \tilde{\boldsymbol{Z}}_t^{\mathrm{T}}\tilde{\boldsymbol{v}}_{-1}\right) \le 0 < T\tilde{q}_t\right\}.$$

Because $\tilde{D}_T(\tilde{\boldsymbol{v}})$ is a piece-wise constant function, which implies that $\tilde{D}_T(\tilde{\boldsymbol{v}})$ takes discrete values in each compact open set, it suffices to show that for any compact set $B \subset \mathbb{R}^{d_1}$ and any $\delta > 0$, there are open neighborhoods $V(\tilde{\boldsymbol{v}}_1), \cdots, V(\tilde{\boldsymbol{v}}_k)$ of some $\tilde{\boldsymbol{v}}_1, \cdots, \tilde{\boldsymbol{v}}_k$ such that $B \subset \cup_{j=1}^{k} V(\tilde{\boldsymbol{v}}_j)$ and

$$\mathbb{P}\left(\cup_{j=1}^{k}\left\{\inf_{\boldsymbol{v}\in V(\tilde{\boldsymbol{v}}_j)}\tilde{D}_T(\boldsymbol{v}) \le \tilde{D}_T(\tilde{\boldsymbol{v}}_j)\right\}\right) < \delta, \tag{B.78}$$

for sufficiently large T.

Let $\{\mathcal{Z}_\phi(\tilde{\boldsymbol{z}}_j), j \le J(\phi)\}$ be $J(\phi)$ closed equal-sized cubes with the side-length ϕ such that $\mathcal{Z}_{-1,1}$, the support of the distribution of $\boldsymbol{Z}_{-1,1}$, can be covered by the union of $\{\mathcal{Z}_\phi(\tilde{\boldsymbol{z}}_j), j \le J(\phi)\}$, and the center of the cube $\mathcal{Z}_\phi(\tilde{\boldsymbol{z}}_j)$ is denoted as $\tilde{\boldsymbol{z}}_j$. Construct $(2m+1)J(\phi)$ sets $\{V_{kj}, l = -m, \cdots, m, j \le J(\phi)\} \subset \mathbb{R}^{d_1}$ as

$$V_{kj} = \{\tilde{\boldsymbol{v}} \in R^{d_1} : \nu_k - \psi < \tilde{\boldsymbol{z}}^{\mathrm{T}} \tilde{\boldsymbol{v}}_{-1} < \nu_k + \psi, \forall \tilde{\boldsymbol{z}} \in \mathcal{Z}_\phi(\tilde{\boldsymbol{z}}_j)\},$$

where $\psi > 0$ and $\nu_k = k\psi$ for $k \in \{-m, \cdots, 0, \cdots, m\}$. Since $\mathcal{Z}_{-1,1}$ is a compact set, which implies that the range of $\tilde{\boldsymbol{z}}^{\mathrm{T}} \tilde{\boldsymbol{v}}_{-1}$ is compact for any compact B, the union of $\{V_{kj}\}$ can cover B by selecting sufficiently large m.

Because $\tilde{D}_T(\tilde{\boldsymbol{v}})$ is piece-wise constant, a discontinuity of $\tilde{D}_T(\tilde{\boldsymbol{v}})$ can potentially occur in $\cup_j V_{kj}$ only if there exist $\boldsymbol{v}_* \in \cup_j V_{kj}$ and $(T\tilde{q}_{t_*}, \tilde{\boldsymbol{Z}}_{t_*})$ for some $t_* \in \{1, \cdots, T\}$ such that $T\tilde{q}_{t_*} = \tilde{\boldsymbol{Z}}_{t_*}^{\mathrm{T}} \boldsymbol{v}_*$, satisfying $\nu_k - \psi \le T\tilde{q}_{t_*} \le \nu_k + \psi$. If there is such $(T\tilde{q}_{t_*}, \tilde{\boldsymbol{Z}}_{t_*})$, we say $\tilde{D}_T(\tilde{\boldsymbol{v}})$ has a breakpoint in $\cup_j V_{kj}$. Define $\mathcal{B}_T = |\{t : 0 < T\tilde{q}_t < \bar{Z}\}|$, where $\bar{Z} = \sup_{\boldsymbol{z} \in \mathcal{Z}_{-1,1}, \boldsymbol{v} \in B} \boldsymbol{z}^{\mathrm{T}} \boldsymbol{v}$, as an upper bound on the number of breakpoint of $\tilde{D}_T(\tilde{\boldsymbol{v}})$ in B, and let $\mathcal{B} = |\{i : \mathcal{J}_i < \bar{Z}\}|$, where $\mathcal{J}_i = \sum_{m=1}^{i} \mathcal{E}_i$ with $\{\mathcal{E}_i\}_{i=1}^{\infty}$ being i.i.d. unit exponentially distributed variables. Because the point process induced by $\{T\tilde{q}_t, t \in \{1, \cdots, T\} : \tilde{q}_t > 0\}$ weakly converges to the point process induced by $\{\mathcal{J}_i\}_{i=1}^{\infty}$ as shown in the proof of Lemma B.2, by the continuous mapping theorem, we have $\mathcal{B}_T \xrightarrow{d} \mathcal{B}$. Therefore, the number of breakpoints $\mathcal{B}_T = O_p(1)$.

We now show the breakpoints are separated, namely, no more than one breakpoint can happen in $\cup_j V_{kj}$ with probability arbitrarily close to one if ψ is sufficiently small. Let A_k to be the event that $\tilde{D}_T(\tilde{\boldsymbol{v}})$ has more than one breakpoint in $\cup_j V_{kj}$. Relabelling $\{T\tilde{q}_t, t \in \{1, \cdots, T\} : \tilde{q}_t > 0\}$ as $\{\mathcal{J}_{iT}\}$ such that $0 < \mathcal{J}_{1T} \le \mathcal{J}_{2T} \le \cdots$. Then, because the point process corresponding to $\{\tilde{q}_t, t \in \{1, \cdots, T\} : \tilde{q}_t > 0\}$ converges weakly to that corresponding to $\{J_i\}_{i=1}^{\infty}$, according to continuous mapping theorem, for any finite $k \le T$,

$$(\mathcal{J}_{1T}, \cdots, \mathcal{J}_{kT}) \xrightarrow{d} (\mathcal{J}_1, \cdots, \mathcal{J}_k). \tag{B.79}$$

Define A_k to be the event that $\tilde{D}_T(\tilde{\boldsymbol{v}})$ has more than two break-points in $\cup_j V_{kj}$. Since $\cup_k A_k$ happens if at least one pair $(\mathcal{J}_{(i-1)T}, \mathcal{J}_{iT})$ for some $i \le \mathcal{B}_T$ satisfying $\mathcal{J}_{iT} - \mathcal{J}_{(i-1)T} < 2\psi$, we have

$$\begin{aligned}
\limsup_{T\to\infty} \mathbb{P}(\cup_k A_k) &\le \limsup_{T\to\infty} \mathbb{P}\left\{\min_{2 \le i \le \mathcal{N}_T} (\mathcal{J}_{iT} - \mathcal{J}_{(i-1)T}) < 2\psi\right\} \\
&\le \limsup_{T\to\infty} \mathbb{P}\left\{\min_{2 \le i \le K} (\mathcal{J}_{iT} - \mathcal{J}_{(i-1)T}) < 2\psi\right\} + \mathbb{P}(\mathcal{B}_T > K) \\
&\overset{(i)}{\le} \mathbb{P}\left\{\min_{2 \le i \le K} (\mathcal{J}_i - \mathcal{J}_{(i-1)}) < 2\psi\right\} + \mathbb{P}(\mathcal{B} > K) \\
&\overset{(ii)}{\le} \delta/2,
\end{aligned} \tag{B.80}$$

where (i) is by (B.79), (ii) is by taking K sufficiently large such that $\mathbb{P}(\mathcal{B} > K) < \delta/4$, and taking ψ sufficiently small such that $\mathbb{P}\left\{\min_{2 \le i \le K} (\mathcal{J}_i - \mathcal{J}_{(i-1)}) < 2\psi\right\} < \delta/4$. The latter is possible since by definition $\mathcal{J}_i - \mathcal{J}_{(i-1)} = \mathcal{E}_{i-1}$ has independent unit exponential distribution. Hence

$$\mathbb{P}\left\{\min_{2 \le i \le K} (\mathcal{J}_i - \mathcal{J}_{(i-1)}) < 2\psi\right\} = \mathbb{P}\left\{\min_{2 \le i \le K} \mathcal{E}_{i-1} < 2\psi\right\} = 1 - e^{-2\psi(K-1)},$$

which converges to 0 as $\psi(K-1) \to 0$.

We construct centers $\tilde{\boldsymbol{v}}_{kj}$ in V_{kj} such that

$$\nu_k - \psi < \tilde{\boldsymbol{z}}^{\mathrm{T}} \tilde{\boldsymbol{v}}_{-1,kj} < \nu_k - \psi + \eta, \ \ \forall \tilde{\boldsymbol{z}} \in \mathcal{Z}_\phi(\tilde{\boldsymbol{z}}_j),$$

where η will be set sufficiently small in the next step. Depending on η, we will set ϕ sufficiently small as well to satisfy the above constraints. Note that the left-side hand of (B.78) can be decomposed as

$$\mathbb{P}\left(\cup_{j,k}\left\{\inf_{\boldsymbol{v}\in V_{kj}(\tilde{\boldsymbol{v}}_{kj})} \tilde{D}(\boldsymbol{v}) \le \tilde{D}_T(\tilde{\boldsymbol{v}}_{kj})\right\}\right) < \limsup_{T\to\infty} \mathbb{P}\{B(\eta)\} + \limsup_{T\to\infty} \mathbb{P}(\cup_k A_k), \quad \text{(B.81)}$$

where $B(\eta)$ is the event that $\{\mathcal{J}_{iT}, i \le K\}$ are separated, and at least one of $\mathcal{J}_{iT} \in [\nu_{k_i} - \psi, \nu_{k_i} - \psi + \eta]$ for some $k_i \in \{1, \cdots, K\}$. The bound (B.81) holds because $\tilde{D}(\boldsymbol{v})$ can only jump if $\mathcal{J}_{iT}$ increases, implying that

$$\cup_{j,k}\left\{\inf_{\boldsymbol{v}\in V_{kj}(\tilde{\boldsymbol{v}}_{kj})} \tilde{D}(\boldsymbol{v}) \le \tilde{D}_T(\tilde{\boldsymbol{v}}_{kj})\right\} \cap (\cup_k A_k)^c = B(\eta).$$

Due to (B.79) and the fact that $\{\mathcal{J}_i\}$ have a bounded density, we have

$$\limsup_{T\to\infty} \mathbb{P}\{B(\eta)\} = O(K\eta) < \delta/2, \quad \text{(B.82)}$$

by choosing η sufficiently small. Combining (B.80)–(B.82) completes the proof for Lemma B.3. □

B.9. Proof of Lemma B.4.

PROOF. Let $f_n(\boldsymbol{v}) = \sum_{i=0}^{\infty} a_{ni} \mathbb{1}(\boldsymbol{v} \in F_{ni})$, where $\{a_{ni} \in \mathbb{R}\}_{i=1}^{\infty}$ are jump sizes and $\{F_{ni} \in \mathbb{R}^d\}_{i=0}^{\infty}$ are non-overlapping level sets. Let $\tilde{f}_n(\boldsymbol{v}) = \sum_{i=0}^{\infty} i \mathbb{1}(\boldsymbol{v} \in F_{ni})$ be the associated jump process. Note $\tilde{f}_n$ has a jump with size 1 at the boundary of each level set F_{ni}. Let the limiting piece-wise constant function be $f_0(\boldsymbol{v}) = \sum_{i=0}^{\infty} a_{0i} \mathbb{1}(\boldsymbol{v} \in F_{0i})$, whose associated jump process be $\tilde{f}_0(\boldsymbol{v}) = \sum_{i=0}^{\infty} i \mathbb{1}(\boldsymbol{v} \in F_{0i})$. For any compact set E, we define $I_n(E) = \{i : F_{n,i} \cap E \notin \emptyset\}$ and $I_0(E) = \{i : F_{0,i} \cap E \notin \emptyset\}$ be the index sets for the level sets of f_n and f_0 that have intersections with E, respectively. Let the argmin sets of f_n and f_0 on the compact set E be G_n and G_0, respectively.

Step 1. Convergence of level sets.

We first show the convergence of the level sets $\{F_{ni}, i \in I_n(E)\}$ to $\{F_{0i}, i \in I_n(E)\}$, using the epi-convergence of the jump processes $\{\tilde{f}_n\}$. For any interior point $\boldsymbol{v}_i$ in $F_{0,i}$, which is a continuous point of f_0 and $\tilde{f}_0$, let $\varepsilon_0 > 0$ be any sufficiently small constant such that $\mathcal{N}(\boldsymbol{v}_i; \varepsilon_0) \subset F_{0,i}$. By a similar argument to that used in the proof of Lemma B.3, there exists some $\boldsymbol{v}_i' \in \mathcal{N}(\boldsymbol{v}_i; \varepsilon_0)$ such that f_n and $\tilde{f}_n$ are asymptotically equi-lower semicontinuous at $\boldsymbol{v}_i'$. Since we have $\{\tilde{f}_n\}$ epi-converge to $\tilde{f}_0$, by applying Theorem 7.10 of Rockafellar and Wets (1998), we have the pointwise convergence $\tilde{f}_n(\boldsymbol{v}_i') \to \tilde{f}_0(\boldsymbol{v}_i')$ as $n \to \infty$. Let $\partial F_{0,i} = \bar{F}_{0,i} \setminus F_{0,i}^{\circ}$ be the boundary of $F_{0,i}$ for each $i \in I_0(E)$, where sets $\bar{F}_{0,i}$ and $F_{0,i}^{\circ}$ are the closure and interior of $F_{0,i}$, respectively. Then we can find infinitely many $\boldsymbol{v} \in F_{0,i}^{\circ}$ with $\inf_{\boldsymbol{v}'\in\partial F_{0,i}} \|\boldsymbol{v} - \boldsymbol{v}'\| = \varepsilon_0$, such that $\tilde{f}_n(\boldsymbol{v}) \to \tilde{f}_0(\boldsymbol{v}) = i$. This together with the connectness of $F_{n,i}$ implies that $F_{0,i}^{\varepsilon_0-} \subset F_{n,i}$, where $F_{0,i}^{\varepsilon_0-} = \bar{F}_{0,i} \setminus \{\boldsymbol{v} \in \bar{F}_{0,i}, \inf_{\boldsymbol{v}'\in\partial F_{0,i}} \|\boldsymbol{v} - \boldsymbol{v}'\| \le \varepsilon_0\}$. Similarly we can find infinitely many $\boldsymbol{v} \in E \setminus \bar{F}_{0,i}$ with $\inf_{\boldsymbol{v}'\in\partial F_{0,i}} \|\boldsymbol{v} - \boldsymbol{v}'\| = \varepsilon_0$, such that $\tilde{f}_n(\boldsymbol{v}) \to \tilde{f}_0(\boldsymbol{v}) \neq i$. It means that for each sufficiently large n, there is a jump of $\tilde{f}_n$ in the region $\{\boldsymbol{v} \in E : \inf_{\boldsymbol{v}'\in\partial F_{0,i}} \|\boldsymbol{v} - \boldsymbol{v}'\| < \varepsilon_0\}$ around the boundary of $F_{0,i}$ for each $i \in I_0(E)$. Therefore, we obtain $|I_n(E)| \to |I_0(E)|$ as $n \to \infty$. Also, since ε_0 can be taken arbitrarily close to 0 and

$\mu(\partial F_{0,i}) = 0$, where μ is the Lebesgue measure, for each $1 \le i \le N_0$ and each sufficiently large n, it holds that $|\mathbb{1}(\boldsymbol{v} \in F_{n,i}) - \mathbb{1}(\boldsymbol{v} \in F_{0,i})| \to 0$ almost surely under the Lebesgue measure.

Step 2. Convergence of the argmin level set.

We now show the minimzed set of f_n converges to that of f_0. Let F_{0,i_*} be the level set on which f_0 attains its minimum. By the condition that $\xi_{0,i} \neq \xi_{0,j}$ if $i \neq j$, such i_* is unique. Hence $F_{0,i_*} = G_0$. Note that unlike the proof of Theorem 1 of Knight (1999), applying Theorem 7.33 of Rockafellar and Wets (1998) can only ensure $G_n \subset G_0$ asymptotically. However, such result can be strengthened by utilizing the piece-wise constant property of f_n and f_0. From the above paragraph, it has been shown that each level set $F_{n,i}$ of f_n converges to $F_{0,i}$ of f_0. As argued in the previous paragraph, for each $i \in I_0(E)$ we can find $\boldsymbol{v}_i \in F_{0,i}$, such that f_0 is continuous at $\boldsymbol{v}_i$ and $\{f_n\}$ are asymptotically equi-lower semicontinuous at $\boldsymbol{v}_i$. Hence the epi-convergence of $\{f_n\}$ to f_0 implies the pointwise convergence of $f_n(\boldsymbol{v}_i) \to f_0(\boldsymbol{v}_i) = a_{i0}$, meaning that $a_{n,i} \to a_{0,i}$ for each $i \in I_0(E)$. Because $\{a_{0,i}, i \in I_0(E)\}$ is uniquely minimized at $i = i_*$, for any $\epsilon > 0$ such that for any sufficiently large n, we have $a_{n,i_*} < a_{n,i} + \epsilon$, which means that the minimizer level set G_n of f_n is unique and equals to $F_{n,i*}$. This together with the result in the previous paragraph implies $|\mathbb{1}(\boldsymbol{v} \in G_n) - \mathbb{1}(\boldsymbol{v} \in G_0)| \to 0$ for almost surely $\boldsymbol{v}$. The desired result (B.27) in Lemma B.4 then follows by applying the dominated convergence theorem. □

B.10. Proof of Lemma B.5.

PROOF. By (B.24), $W_T(\boldsymbol{u})$ can be written as $W_T(\boldsymbol{u}) = \sum_{i=1}^4 (\boldsymbol{u}_i^{\mathrm{T}} \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}} \mathbb{1}\{\boldsymbol{Z} \in R_i(\boldsymbol{\gamma}_0)\}]\boldsymbol{u}_i - 2\boldsymbol{u}_i^{\mathrm{T}} H_{i,T})$, where

$$H_{i,T} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \boldsymbol{X}_t \varepsilon_t \mathbb{1}\{\boldsymbol{Z}_t \in R_i(\boldsymbol{\gamma}_0)\}.$$

Let $\boldsymbol{a} \in \mathbb{R}^p$ with $\|\boldsymbol{a}\| = 1$. Also let $H_{\boldsymbol{a},i,T} = \boldsymbol{a}^{\mathrm{T}} H_{i,T}$ and $\sigma^2_{\boldsymbol{a},i} = \boldsymbol{a}^{\mathrm{T}} \Sigma_i \boldsymbol{a}^{\mathrm{T}}$, where $\Sigma_i = \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}} \varepsilon^2 \mathbb{1}\{\boldsymbol{Z} \in R_i(\boldsymbol{\gamma}_0)\}]$. Then with Assumptions 1.(ii) and 3.(i), by the martingale central limit theorem (Hall and Heyde, 1980), it holds that $\sigma^{-1}_{\boldsymbol{a},i} H_{\boldsymbol{a},i,T} \xrightarrow{d} N(0,1)$. Hence, by the Cramer-Wold device, we obtain $H_{i,T} \xrightarrow{d} N(0, \Sigma_i)$, which implies that $W_T(\boldsymbol{u}) \xrightarrow{d} W(\boldsymbol{u})$. Since the stochastic component of $W_T(\boldsymbol{u})$ is linear in $\boldsymbol{u}$, the stochastic equicontinuity of $W_T(\boldsymbol{u})$ can be trivially proved. Hence $W_T \xrightarrow{d} W$ in $\ell^\infty(\mathbb{B})$.

We now show the asymptotic independence between $W_T(\boldsymbol{u})$ and $D_T(\boldsymbol{v})$. For independence observations, it can be readily proved by the characteristic function approach used in Yu (2012), which however may not be suitable for the dependence case. In this proof, we employ the device established in Hsing (1995), which can be used to show the asymptotic independence between the extreme type and sum type statistics for the mixing sequences. We notice that while the original results in that paper were for univariate random variables, they can be extended to multivariate cases with essentially the same proof.

As in Part 1 of the proof of Lemma B.2, we write $D_T(\boldsymbol{v}) = \sum_{l=1}^2 \sum_{(j,k) \in \mathcal{S}(l)} \mathcal{T}^{(j,k)}_{l,\boldsymbol{v}_l} \left(\widehat{\mathbf{N}}^{(j,k)}_{l,T} \right)$, where $\widehat{\mathbf{N}}^{(j,k)}_{l,T}$ is a point process defined in (B.46) and $\mathcal{T}^{(j,k)}_{l,\boldsymbol{v}_l}$ is a continuous functional. Therefore, it suffices to show the asymptotic independence between $\widehat{\mathbf{N}}^{(j,k)}_{l,T}$ and $H_{\boldsymbol{a},i,T}$ for any $\boldsymbol{a} \in \mathbb{R}^p$ with $\|\boldsymbol{a}\| = 1$, $l \in \{1,2\}$, $(j,k) \in \mathcal{S}(l)$ and $i \in \{1, \cdots, 4\}$. If one has

$$\mathbb{P}\left\{ H_{\boldsymbol{a},i,T}/\sigma_{\boldsymbol{a},i,t} \le x, \widehat{\mathbf{N}}^{(j,k)}_{l,T}(F_i) = k_i, 1 \le i \le s \right\}$$

$$=\mathbb{P}\left(H_{\boldsymbol{a},i,T}/\sigma_{\boldsymbol{a},i,t}\le x\right)\mathbb{P}\left\{\widehat{\mathbf{N}}_{l,T}^{(j,k)}(F_i)=k_i,1\le i\le s\right\}+o(1), \tag{B.83}$$

for any $x\in\mathbb{R}$, positive integer s, non-negative integers $\{k_i\}_{i=1}^s$, and non-overlapping sets $\{F_i=(F_{1i},F_{2i},F_{3i})\}_{i=1}^s\in\mathcal{E}(l)$, where $\mathcal{E}(l)$ is the basis of relatively compact open set in E_l as used in Part 2 of Section B.7, then $\widehat{\mathbf{N}}_{l,T}^{(j,k)}$ is independent with $H_{\boldsymbol{a},i,T}$.

First, we verify Conditions (2.1) and (2.2) of Hsing (1995). Let $\zeta_t=(Tq_{l,t},\boldsymbol{Z}_{-1,l,t},\zeta_t^{(j,k)})$ and $B_{T,i}=F_i$ for $1\le i\le s$ and $B_{T,s+1}=\cap_{i=1}^s B_{T,i}^c$. Then,

$$\begin{aligned}\limsup_{T\to\infty}T\mathbb{P}(\zeta_t\notin B_{T,s+1})\le&\limsup_{T\to\infty}\sum_{i=1}^s T\mathbb{P}(\zeta_t\in B_{T,i})\\=&\sum_{i=1}^s\mu_l^{(j,k)}(B_{T,i})<\infty,\end{aligned} \tag{B.84}$$

where the equality is due to (B.49). Hence, Condition (2.1) of Hsing (1995) is ensured. In addition, Condition (2.2) of the same paper also holds, since

$$\begin{aligned}&\lim_{l\to\infty}\limsup_{T\to\infty}\mathbb{P}\left\{\cup_{t=l}^T(\zeta_t\notin B_{T,s+1})|\zeta_1\notin B_{T,s+1}\right\}\\=&\lim_{l\to\infty}\limsup_{T\to\infty}\frac{\mathbb{P}\left\{\cup_{t=l}^T(\zeta_t\notin B_{T,s+1})\cap(\zeta_1\notin B_{T,s+1})\right\}}{\mathbb{P}(\zeta_1\notin B_{T,s+1})}\\=&\lim_{l\to\infty}\limsup_{T\to\infty}\frac{O(T^{-2})}{O(T^{-1})}=0,\end{aligned}$$

where the denominator part is from (B.84) and the numerator is derived in the same way as in (B.51).

We now show the desired (B.83) with similar arguments as in Theorem 2.2 of Hsing (1995). Let $\tilde{\zeta}_T=(\zeta_1,\cdots,\zeta_T)^{\mathrm{T}}$. For any $\tilde{A}=(A_1,\cdots,A_T)$, the notation $\tilde{\zeta}_T\in\tilde{A}$ stands for $\zeta_t\in A_t$ for each $1\le t\le T$, and $\tilde{\zeta}_T\notin\tilde{A}$ otherwise. Let $\tilde{\mathcal{B}}_T=\{\tilde{B}=(B_1,\cdots,B_T)\}$, where each $B_t\in\{B_{T,1},\cdots,B_{T,s+1}\}$ for each $1\le t\le T$. Also we let

$$\tilde{\mathcal{B}}_T'=\left\{\tilde{B}\in\tilde{\mathcal{B}}_T:\sum_{t=1}^T\mathbb{1}(B_t=B_{T,i})=k_i,\ \text{ for }1\le i\le s\right\}.$$

By such constructions, we have

$$\begin{aligned}&\left\{\cup_{\tilde{B}\in\tilde{\mathcal{B}}}(\tilde{\zeta}_T\in\tilde{B})\right\}\cap\left\{\widehat{\mathbf{N}}_{l,T}^{(j,k)}(B_{T,i})=k_i,1\le i\le s\right\}\\\overset{(i)}{=}&\cup_{\tilde{B}\in\tilde{\mathcal{B}}'}(\tilde{\zeta}_T\in\tilde{B})\overset{(ii)}{=}\left\{\widehat{\mathbf{N}}_{l,T}^{(j,k)}(B_{T,i})=k_i,1\le i\le s\right\}.\end{aligned}$$

Also, we note that (i) implies that

$$\begin{aligned}0\le&\mathbb{P}\left\{H_{\boldsymbol{a},i,T}/\sigma_{\boldsymbol{a},i,t}\le x,\widehat{\mathbf{N}}_{l,T}^{(j,k)}(F_i)=k_i,1\le i\le s\right\}\\&-\mathbb{P}\left\{H_{\boldsymbol{a},i,T}/\sigma_{\boldsymbol{a},i,t}\le x,\cup_{\tilde{B}\in\tilde{\mathcal{B}}'}(\tilde{\zeta}_T\in\tilde{B})\right\}\\\le&\mathbb{P}\left\{\cap_{\tilde{B}\in\tilde{\mathcal{B}}}(\tilde{\zeta}_T\notin\tilde{B})\right\}.\end{aligned} \tag{B.85}$$

With the fact that the events $\{(\tilde{\zeta}_T\notin\tilde{B})\}_{\tilde{B}\in\tilde{\mathcal{B}}'}$ are disjoint, repeatedly applying Theorem 2.1 of Hsing (1995) leads to

$$\mathbb{P}\left\{H_{\boldsymbol{a},i,T}/\sigma_{\boldsymbol{a},i,t}\le x,\widehat{\mathbf{N}}_{l,T}^{(j,k)}(F_i)=k_i,1\le i\le s\right\}$$

$$\overset{(iii)}{=}\sum_{\tilde{B}\in\tilde{\mathcal{B}}'}\mathbb{P}\left\{H_{\boldsymbol{a},i,T}/\sigma_{\boldsymbol{a},i,t}\le x,\tilde{\zeta}_T\in\tilde{B}\right\}+o(1)$$

$$\overset{(iv)}{=}\mathbb{P}(H_{\boldsymbol{a},i,T}/\sigma_{\boldsymbol{a},i,t}\le x)\sum_{\tilde{B}\in\tilde{\mathcal{B}}'}\mathbb{P}\left(\tilde{\zeta}_T\in\tilde{B}\right)+o(1)$$

$$=\mathbb{P}(H_{\boldsymbol{a},i,T}/\sigma_{\boldsymbol{a},i,t}\le x)\mathbb{P}\left\{\cup_{\tilde{B}\in\tilde{\mathcal{B}}'}(\tilde{\zeta}_T\in\tilde{B})\right\}+o(1)$$

$$\overset{(v)}{=}\mathbb{P}(H_{\boldsymbol{a},i,T}/\sigma_{\boldsymbol{a},i,t}\le x)\mathbb{P}\left\{\widehat{\mathbf{N}}_{l,T}^{(j,k)}(F_i)=k_i,1\le i\le s\right\}+o(1),$$

where (iii) is because of (B.85) and (2.3) in Theorem 2.1 of Hsing (1995), (iv) is implied by (2.4) in the same theorem, and (v) is due to the equivalence relationship (ii). Hence, (B.83) is now verified. Since the above derivations hold for any $\boldsymbol{a}\in\mathbb{R}^p$ with $\|\boldsymbol{a}\|=1$, $l\in\{1,2\},(j,k)\in\mathcal{S}(l)$ and $i\in\{1,\cdots,4\}$, we complete the proof for the asymptotic independence between $W_T(\boldsymbol{u})$ and $D_T(\boldsymbol{v})$. □

APPENDIX C: PROOF FOR SECTION 4 AND ADDITIONAL ALGORITHMS

C.1. Proof of Theorem 4.1. The following proof is for Theorem 4.1 on the validity of the MIQP.

PROOF. Let the criterion function of the MIQP be

$$\mathbb{V}_T(\boldsymbol{\ell})=\frac{1}{T}\sum_{t=1}^{T}\left(Y_t-\sum_{k=1}^{4}\sum_{i=1}^{p}X_{t,i}\ell_{k,i,t}\right)^2,$$

where $\boldsymbol{\ell}=\{\ell_{k,i,t}:k=1,\cdots,4,i=1,\cdots,p,t=1,\cdots,T\}$. The constraints of the MIQP are

1. $\boldsymbol{\beta}_k\in\mathcal{B},\quad\boldsymbol{\gamma}_j\in\Gamma,$
2. $g_{j,t}\in\{0,1\},\quad I_{k,t}\in\{0,1\},$
3. $L_i\le\beta_{k,i}\le U_i,$
4. $(g_{j,t}-1)(M_{j,t}+\epsilon)<\boldsymbol{Z}_{j,t}^{\mathrm{T}}\boldsymbol{\gamma}_j\le g_{j,t}M_{j,t},$
5.(i). $I_{k,t}L_i\le\ell_{k,i,t}\le I_{k,t}U_i,$
5.(ii). $L_i(1-I_{k,t})\le\beta_{k,i}-\ell_{k,i,t}\le U_i(1-I_{k,t}),$
6. $I_{k,t}\le s_j^{(k)}g_{j,t}+(1-s_j^{(k)})/2,\ \ I_{k,t}\ge\sum_{j=1}^{2}\left\{s_j^{(k)}g_{j,t}+(1-s_j^{(k)})/2\right\}-1,$

for $k=1,\cdots,4,j=1,2,i=1,\cdots,p$ and $t=1,\cdots,T$. Define $\boldsymbol{g}=\{g_{j,t}:j=1,2,t=1,\cdots,T\}$, $\boldsymbol{\mathcal{I}}=\{I_{k,t}:k=1,\cdots,4,t=1,\cdots,T\}$. The solution of the MIQP is denoted as $(\bar{\boldsymbol{\beta}},\bar{\boldsymbol{\gamma}},\bar{\boldsymbol{g}},\bar{\boldsymbol{\mathcal{I}}},\bar{\boldsymbol{\ell}})=\arg\min_{\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{g},\boldsymbol{\mathcal{I}},\boldsymbol{\ell}}\mathbb{V}_T(\boldsymbol{\ell})$.

To prove the theorem, it suffices to show that (i) $\mathbb{M}_T(\bar{\boldsymbol{\theta}})=\mathbb{V}_T(\bar{\boldsymbol{\ell}})$, where $\bar{\boldsymbol{\theta}}=(\bar{\boldsymbol{\gamma}}^{\mathrm{T}},\bar{\boldsymbol{\beta}}^{\mathrm{T}})^{\mathrm{T}}$; (ii) $\mathbb{V}_T(\bar{\boldsymbol{\ell}})\ge\mathbb{M}_T(\widehat{\boldsymbol{\theta}})$; and (iii) $\mathbb{M}_T(\widehat{\boldsymbol{\theta}})\ge\mathbb{V}_T(\bar{\boldsymbol{\ell}})$.

Proof of (i): It is sufficient to show that

$$\left(Y_t-\sum_{k=1}^{4}\boldsymbol{X}_t^{\mathrm{T}}\bar{\boldsymbol{\beta}}_k\mathbb{1}_j(\boldsymbol{Z}_{1,t}^{\mathrm{T}}\bar{\boldsymbol{\gamma}}_1,\boldsymbol{Z}_{2,t}^{\mathrm{T}}\bar{\boldsymbol{\gamma}}_2)\right)^2=\left(Y_t-\sum_{k=1}^{4}\sum_{i=1}^{p}X_{t,i}\bar{\ell}_{k,i,t}\right)^2.\qquad\text{(C.1)}$$

We show that $\bar{\ell}_{k,i,t} = \bar{\beta}_{k,i}\mathbb{1}_k(\boldsymbol{Z}_{1,t}^{\mathrm{T}}\bar{\boldsymbol{\gamma}}_1, \boldsymbol{Z}_{2,t}^{\mathrm{T}}\bar{\boldsymbol{\gamma}}_2)$. If $\mathbb{1}_k(\boldsymbol{Z}_{1,t}^{\mathrm{T}}\bar{\boldsymbol{\gamma}}_1, \boldsymbol{Z}_{2,t}^{\mathrm{T}}\bar{\boldsymbol{\gamma}}_2) = 1$, then by Constraints 2 and 6, we have $I_{k,t} = 1$, which implies that $\bar{\ell}_{k,i,t} = \bar{\beta}_{k,i}$. If $\mathbb{1}_k(\boldsymbol{Z}_{1,t}^{\mathrm{T}}\bar{\boldsymbol{\gamma}}_1, \boldsymbol{Z}_{2,t}^{\mathrm{T}}\bar{\boldsymbol{\gamma}}_2) = 0$, then by Constraints 2 and 6 we have $I_{k,t} = 0$, which implies that $\bar{\ell}_{k,i,t} = 0$. Combining the two cases verifies $\bar{\ell}_{k,i,t} = \bar{\beta}_{k,i}\mathbb{1}_k(\boldsymbol{Z}_{1,t}^{\mathrm{T}}\bar{\boldsymbol{\gamma}}_1, \boldsymbol{Z}_{2,t}^{\mathrm{T}}\bar{\boldsymbol{\gamma}}_2)$ for each k, i, t, which implies (C.1).

Proof of (ii): Note that

$$V_T(\bar{\boldsymbol{\ell}}) = \mathbb{M}_T(\bar{\boldsymbol{\theta}}) \geq \min_{\boldsymbol{\beta}\in\mathcal{A},\boldsymbol{\gamma}\in\mathcal{G}} \mathbb{M}_T(\boldsymbol{\theta}) = \mathbb{M}_T(\widehat{\boldsymbol{\theta}}),$$

where the first equality is by (i) and the last equality is by the definition of $\widehat{\boldsymbol{\theta}}$.

Proof of (iii): Define $\widehat{\ell}_{k,i,t} = \widehat{\beta}_{k,i}\widehat{I}_{k,t}$, where $\widehat{I}_{k,t} = \prod_{j=1}^{2} s_j^{(k)}\widehat{g}_{j,t}$ and $\widehat{g}_{j,t} = \mathbb{1}\{\boldsymbol{Z}_t^{\mathrm{T}}\widehat{\boldsymbol{\gamma}}_j > 0\}$. Then by definition $\mathbb{M}_T(\widehat{\boldsymbol{\theta}}) = \mathbb{V}_T(\widehat{\boldsymbol{\ell}})$, where $\widehat{\boldsymbol{\ell}} = \{\widehat{\ell}_{k,i,t}\}$. If $(\widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{d}}, \widehat{\boldsymbol{\ell}})$ satisfy Constraints 1-6 above, then by the definition of $\bar{\boldsymbol{\ell}}$, we have $\mathbb{V}_T(\widehat{\boldsymbol{\ell}}) \geq \mathbb{V}_T(\bar{\boldsymbol{\ell}})$ and hence, (iii) can be verified. Constraints 1-3 are ensured by the definitions. For Constraint 4, note that if $\boldsymbol{Z}_{j,t}^{\mathrm{T}}\widehat{\boldsymbol{\gamma}}_j > 0$, then by definition $\widehat{g}_{j,t} = I(\boldsymbol{Z}_{j,t}^{\mathrm{T}}\widehat{\boldsymbol{\gamma}}_j > 0) = 1$. Constraint 4 becomes $0 < \boldsymbol{Z}_{j,t}^{\mathrm{T}}\widehat{\boldsymbol{\gamma}}_j \leq M_{j,T} = \sup_{\boldsymbol{\gamma}\in\Gamma_j} |\boldsymbol{Z}_{j,t}^{\mathrm{T}}\boldsymbol{\gamma}|$, which is satisfied. When $\boldsymbol{Z}_{j,t}^{\mathrm{T}}\widehat{\boldsymbol{\gamma}}_j \leq 0$, then $\widehat{g}_{j,t} = 0$. Condition 4 becomes $-M_{j,t} - \epsilon < \boldsymbol{Z}_{j,t}^{\mathrm{T}}\widehat{\boldsymbol{\gamma}}_j \leq 0$, which holds for any $\epsilon > 0$. Hence, Condition 4 is verified. For Condition 5, note that if $\widehat{I}_{k,t} = 1$, then $\widehat{\ell}_{k,i,t} = \widehat{\beta}_{k,i}$ by its definition, which meet the requirement in Constraint 5 (i) and (ii). Otherwise, if $\widehat{I}_{k,t} = 0$, then $\widehat{\ell}_{k,i,t} = 0$, and Constraints 5 (i) and (ii) are satisfied. For Constraint 6, it is ready to verify that

$$\sum_{j=1}^{2}\left\{s_j^{(k)}\widehat{g}_{j,t} + (1 - s_j^{(k)})/2\right\} - 1 \leq \prod_{j=1}^{2} s_j^{(k)}\widehat{g}_{j,t} \leq s_j^{(k)}\widehat{g}_{j,t} + (1 - s_j^{(k)})/2,$$

for any $\widehat{g}_{1,t}, \widehat{g}_{2,t} \in \{0, 1\}$ and $s_1^{(k)}, s_2^{(k)} \in \{-1, 1\}$. In summary, $(\widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{d}}, \widehat{\boldsymbol{\ell}})$ satisfies Constraints 1-6, implying that

$$\mathbb{M}_T(\widehat{\boldsymbol{\theta}}) = \mathbb{V}_T(\widehat{\boldsymbol{\ell}}) \geq \mathbb{V}_T(\bar{\boldsymbol{\ell}}),$$

which proves (iii). Combining parts (i), (ii) and (iii), we obtain $\mathbb{M}_T(\widehat{\boldsymbol{\theta}}) = \mathbb{M}_T(\bar{\boldsymbol{\theta}})$, which completes the proof of Theorem 4.1. □

C.2. Block coordinate descent. The MIQP presented in Section 4 of the main paper may be slow when the dimension of $\boldsymbol{X}_t$ and the sample size T are large. As an alternative, we present a block coordinate descent (BCD) algorithm.

Iterate the following two steps until $\max_{1\leq k\leq 4} \|\widehat{\boldsymbol{\beta}}_k^{s+1} - \widehat{\boldsymbol{\beta}}_k^{s}\| < \eta$.
Step 1. For each given $\widehat{\boldsymbol{\beta}}^s$, solve the following mixed integer linear programming (MILP) problem:

$$\min_{\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{g},\boldsymbol{\mathcal{I}},\boldsymbol{\ell}} \frac{1}{T}\sum_{t=1}^{T}\sum_{k=1}^{4}\left\{(\boldsymbol{X}_t^{\mathrm{T}}\widehat{\boldsymbol{\beta}}_k^s)^2 - 2Y_t\boldsymbol{X}_t^{\mathrm{T}}\widehat{\boldsymbol{\beta}}_k^s\right\} I_{k,t} \tag{C.2}$$

$$\text{subject to } \begin{cases} \boldsymbol{\gamma}_j \in \Gamma_j,\ g_{j,t} \in \{0,1\},\ I_{k,t} \in \{0,1\}; \\ (g_{j,t} - 1)(M_{j,t} + \epsilon) < \boldsymbol{Z}_{j,t}^{\mathrm{T}}\boldsymbol{\gamma}_k \leq g_{j,t}M_{j,t},\ I_{k,t}L_i \leq \ell_{k,i,t} \leq I_{k,t}U_i; \\ I_{k,t} \leq s_l^{(k)}g_{l,t} + \dfrac{1 - s_l^{(k)}}{2},\ I_{k,t} \geq \displaystyle\sum_{l=1}^{2}\left(s_l^{(k)}g_{l,t} + \frac{1 - s_l^{(k)}}{2}\right) + 1 - L, \end{cases} \tag{C.3}$$

for $k = 1, \cdots, 4, j = 1, 2, i = 1, \cdots, d_x$ and $t = 1, \cdots, T$. Let the solution be $\widehat{\boldsymbol{\gamma}}^{s+1}$.

Step 2. For the given $\widehat{\boldsymbol{\gamma}}^{s+1}$, obtain

$$\widehat{\boldsymbol{\beta}}_k^{s+1} = [\mathbb{E}_T\{\boldsymbol{X}_t\boldsymbol{X}_t^{\mathrm{T}}\mathbb{1}(\boldsymbol{Z}_t \in R_k(\widehat{\boldsymbol{\gamma}}^{s+1})\}]^{-1}\mathbb{E}_T\{Y_t\boldsymbol{X}_t\mathbb{1}(\boldsymbol{Z}_t \in R_k(\widehat{\boldsymbol{\gamma}}^{s+1})\}.$$

REMARK C.1. The advantages of the BCD compared with the MIQP are that the optimization with respect to $\boldsymbol{\gamma}$ in each iteration is a linear programming instead of quadratic programming, and that for $\boldsymbol{\beta}_k$ in each iteration has a close form solution. Therefore, the BCD can significantly reduce computation cost. However, unlike the MIQP presented in the main paper, there is no theoretical guarantee for the global optimality of the solutions of the BCD. For the BCD, the specification of the initial value $\widehat{\boldsymbol{\theta}}^0$ is important. In practice, it can be obtained from a grid search procedure, or we can use the output of the MIQP after several iterations as the initial value for the BCD.

The following Table S1 reports the comparison between the joint MIQP algorithm proposed in Section 4 of the main paper and the block coordinate descent algorithm presented in Section C.2. The sample was generated according to

$$Y_t = \sum_{k=1}^{4} \boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\beta}_{k0}\mathbb{1}_k(\boldsymbol{Z}_{1,t}^{\mathrm{T}}\boldsymbol{\gamma}_{10}, \boldsymbol{Z}_{2,t}^{\mathrm{T}}\boldsymbol{\gamma}_{20}) + \varepsilon_t \quad t = 1, \cdots, T$$

where $\boldsymbol{X}_t = (\tilde{\boldsymbol{X}}_t^{\mathrm{T}}, 1)^{\mathrm{T}}$ with $\tilde{\boldsymbol{X}}_t = (X_{1,t}, \cdots, X_{p-1,t})^{\mathrm{T}}$ and $\boldsymbol{Z}_{j,t} = (\tilde{\boldsymbol{Z}}_{j,t}^{\mathrm{T}}, 1)^{\mathrm{T}}$ with $\tilde{\boldsymbol{Z}}_{j,t} = (Z_{j,1,t}, \cdots, Z_{j,d-1,t})^{\mathrm{T}}$ for $j = 1, 2$, and the residuals $\varepsilon_t = \sigma(\boldsymbol{X}_t, \boldsymbol{Z}_t)e_t$ with $\sigma(\boldsymbol{X}_t, \boldsymbol{Z}_t) = 1 + 0.1X_{1,t}^2 + 0.1Z_{1,1,t}^2$ and $\{e_t\}_{t=1}^T$ being generated independently from the standard normal distribution and being independent of $\{\boldsymbol{X}_t, \boldsymbol{Z}_t\}_{t=1}^T$. Let $\boldsymbol{V}_t = (\tilde{\boldsymbol{X}}_t^{\mathrm{T}}, \tilde{\boldsymbol{Z}}_{1,t}^{\mathrm{T}}, \tilde{\boldsymbol{Z}}_{2,t}^{\mathrm{T}})^{\mathrm{T}}$. We generated $\{\boldsymbol{V}_t\}_{t=1}^T \overset{\text{i.i.d.}}{\sim} \boldsymbol{N}(\boldsymbol{0}, \Sigma_V)$, where $\Sigma_V = (\sigma_{ij})_{i,j=1,\cdots,7}$ with $\sigma_{ii} = 1$ and $\sigma_{ij} = 0.1$ if $i \neq j$. We considered two sets of dimensions for $\boldsymbol{X}_t$ and $\boldsymbol{Z}_t$. For $p = 4$ and $d = 3$, the regression coefficients of the four regimes were $\boldsymbol{\beta}_{10} = (1, 1, 1, 1)^{\mathrm{T}}, \boldsymbol{\beta}_{20} = (-3, -2, -1, 0), \boldsymbol{\beta}_{30} = (0, 1, 3, -1)^{\mathrm{T}}$ and $\boldsymbol{\beta}_{40} = (2, -1, 0, 2)^{\mathrm{T}}$, and the two boundary coefficients $\boldsymbol{\gamma}_{10} = (1, -1, 0)^{\mathrm{T}}$ and $\boldsymbol{\gamma}_{20} = (1, 1, 0)^{\mathrm{T}}$, respectively. For $p = 10$ and $d = 6$, the regression coefficients of the four regimes were $\boldsymbol{\beta}_{10} = (1, 1, 1, 1, 1, 0, \cdots, 0)^{\mathrm{T}}, \boldsymbol{\beta}_{20} = (-3, -2, -1, 1, 0, \cdots, 0), \boldsymbol{\beta}_{30} = (0, 1, 3, -1, 1, 0, \cdots, 0)^{\mathrm{T}}$ and $\boldsymbol{\beta}_{40} = (2, -1, 0, 2, 1, 0, \cdots, 0)^{\mathrm{T}}$, and the two boundary coefficients $\boldsymbol{\gamma}_{10} = (1, -1, -1, -1, 0, 0)^{\mathrm{T}}$ and $\boldsymbol{\gamma}_{20} = (1, 1, 1, 1, 0, 0)^{\mathrm{T}}$, respectively. The simulation experimented four sample sizes: $\{200, 400, 800, 1600\}$, and the experiments were repeated 500 times for each sample size. The initial values for the BCD were set as the outputs of the MIQP after $5\log(T)$ iterations. The stopping criterion parameter was specified as $\eta = 10^{-4}$.

Table S1 shows that the estimation errors of both $\boldsymbol{\gamma}_0$ and $\boldsymbol{\beta}_0$ obtained with the BCD were slightly larger than those with the MIQP, while their discrepancies were shrinking as T increased. The running time of the BCD, on the other hand, was significantly shorter than that of the joint MIQP, especially when the dimensions and sample sizes were large, because of the reasons we discussed above and in the main paper. Therefore, it is advocated to use the iterative BCD for large dimensions and sample sizes. However, it should also be noted that it is crucial to choose a good initial value for the BCD for its success. In the above simulations, we used the outputs of the MIQP after several iterations to ensure the quality of the initial values, as poor initial values can lead to large estimation errors.

C.3. MIQP for the three-regime models. The MIQP algorithms are not only suitable for solving the LS problem of the four-regime segmented regression but can also be extended to other segmented regressions. In Section 7.2 of the main paper we have reported simulation

TABLE S1

Empirical average estimation errors $\|\boldsymbol{\gamma}_0 - \widehat{\boldsymbol{\gamma}}\|_2$ and $\|\boldsymbol{\beta}_0 - \widehat{\boldsymbol{\beta}}\|_2$ (multiplied by 10), and running time (in second) obtained with the joint MIQP algorithm and the block coordinate descent (BCD) algorithm. The numbers inside the parentheses are the standard errors of the simulated averages.

	$p=4, d=3$						$p=10, d=6$					
T	MIQP			BCD			MIQP			BCD		
	$\widehat{\boldsymbol{\gamma}}$	$\widehat{\boldsymbol{\beta}}$	Time	$\widehat{\boldsymbol{\gamma}}$	$\widehat{\boldsymbol{\beta}}$	Time	$\widehat{\boldsymbol{\gamma}}$	$\widehat{\boldsymbol{\beta}}$	Time	$\widehat{\boldsymbol{\gamma}}$	$\widehat{\boldsymbol{\beta}}$	Time
200	1.00	6.02	65.3	1.13	6.03	7.7	3.15	10.93	149.1	3.31	11.17	10.0
	(0.59)	(1.15)	(11.3)	(0.62)	(1.14)	(1.2)	(1.83)	(2.40)	(15.3)	(1.88)	(2.49)	(1.9)
400	0.51	4.08	364.1	0.59	4.11	14.3	1.59	7.78	583.6	1.66	7.85	23.9
	(0.31)	(0.76)	(27.9)	(0.28)	(0.79)	(3.1)	(0.92)	(1.51)	(40.1)	(0.98)	(1.63)	(6.4)
800	0.25	2.84	1157.7	0.27	2.85	48.9	0.78	5.20	1817.3	0.82	5.31	69.0
	(0.15)	(0.49)	(53.2)	(0.14)	(0.47)	(7.0)	(0.41)	(0.82)	(89.4)	(0.43)	(0.84)	(11.4)
1600	0.13	2.00	2792.2	0.14	2.00	162.4	0.38	3.61	4502.9	0.40	3.67	208.1
	(0.07)	(0.37)	(162.0)	(0.07)	(0.38)	(12.1)	(0.19)	(0.57)	(217.5)	(0.18)	(0.58)	(21.4)

results under segmented models with less than four regimes to compare the four-regime estimation under misspecifications with the estimation with correctly specified models, where the corresponding MIQPs for less than four regimes models were applied. In this subsection, we present MIQP formulations for the three-regime models with and without intersections. The MIQP for the two-regime model was proposed in Lee et al. (2021).

(i) MIQP for the three-regime model with non-intersected boundaries.

Let $\boldsymbol{g} = \{g_{j,t} : j = 1,2, t = 1, \cdots, T\}$, $\boldsymbol{\mathcal{I}} = \{I_{k,t} : k = 1,2,3, t = 1, \cdots, T\}$ and $\boldsymbol{\ell} = \{\ell_{k,i,t} : k = 1,2,3, i = 1, \cdots, p, t = 1, \cdots, T\}$. Consider solving the following problem

$$\min_{\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{g},\boldsymbol{\mathcal{I}},\boldsymbol{\ell}} \frac{1}{T}\sum_{t=1}^{T}\left(Y_t - \sum_{k=1}^{3}\sum_{i=1}^{p} X_{i,t}\ell_{k,i,t}\right)^2,$$

$$\text{subject to } \begin{cases} \boldsymbol{\beta}_k \in \mathcal{B}, \quad \boldsymbol{\gamma}_j \in \Gamma, \quad g_{j,t} \in \{0,1\}, \quad I_{k,t} \in \{0,1\}, \quad L_i \le \beta_{k,i} \le U_i; \\ (g_{j,t}-1)(M_{j,t}+\epsilon) < \boldsymbol{Z}_{j,t}^{\mathrm{T}}\boldsymbol{\gamma}_j \le g_{j,t}M_{j,t}; \\ g_{j,t}L_i \le \ell_{j,i,t} \le g_{j,t}U_i, \quad I_{2,t}L_i \le \ell_{2,i,t} \le I_{2,t}U_i; \\ L_i(1-g_{j,t}) \le \beta_{k,i} - \ell_{j,i,t} \le U_i(1-g_{j,t}); \\ L_i(1-I_{2,t}) \le \beta_{2,i} - \ell_{2,i,t} \le U_i(1-I_{2,t}); \\ I_{2,t} \le g_{1,t}, \quad I_{2,t} \le 1-g_{2,t}, \quad I_{2,t} \ge g_{1,t} - g_{2,t}, \end{cases}$$

for $k = 1,2,3, j = 1,2, i = 1, \cdots, p$ and $t = 1, \cdots, T$.

(ii) MIQP for the three-regime model with intersected boundaries.

Let $\boldsymbol{g} = \{g_{j,t} : j = 1,2, t = 1, \cdots, T\}$, $\boldsymbol{\mathcal{I}} = \{I_{k,t} : k = 1,2,3, t = 1, \cdots, T\}$ and $\boldsymbol{\ell} = \{\ell_{k,i,t} : k = 1, \cdots, 3, i = 1, \cdots, p, t = 1, \cdots, T\}$. Solve the following problem:

$$\min_{\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{g},\boldsymbol{\mathcal{I}},\boldsymbol{\ell}} \frac{1}{T}\sum_{t=1}^{T}\left(Y_t - \sum_{k=1}^{3}\sum_{i=1}^{p} X_{i,t}\ell_{k,i,t}\right)^2, \tag{C.4}$$

$$\text{subject to } \begin{cases} \boldsymbol{\beta}_k \in \mathcal{B}, \quad \boldsymbol{\gamma}_j \in \Gamma; \\ g_{j,t} \in \{0,1\}, \quad I_{k,t} \in \{0,1\}; \\ L_i \le \beta_{k,i} \le U_i \\ (g_{j,t}-1)(M_{j,t}+\epsilon) < \boldsymbol{Z}_{j,t}^{\mathrm{T}}\boldsymbol{\gamma}_j \le g_{j,t}M_{j,t}; \\ g_{1,t}L_i \le \ell_{1,i,t} \le g_{1,t}U_i, \quad I_{k,t}L_i \le \ell_{k,i,t} \le I_{k,t}U_i \\ L_i(1-g_{1,t}) \le \beta_{k,i} - \ell_{1,i,t} \le U_i(1-g_{1,t}); \\ L_i(1-I_{k,t}) \le \beta_{k,i} - \ell_{k,i,t} \le U_i(1-I_{k,t}); \\ I_{2,t} \le 1-g_{1,t}, \quad I_{2,t} \le 1-g_{2,t}, \quad I_{2,t} \ge 1-g_{1,t}-g_{2,t}; \\ I_{3,t} \le g_{1,t}, \quad I_{3,t} \le 1-g_{2,t}, \quad I_{3,t} \ge g_{1,t}-g_{2,t}, \end{cases}$$

for $k = 1,2,3, j = 1,2, i = 1,\cdots,p$ and $t = 1,\cdots,T$.

APPENDIX D: PROOFS FOR SECTION 5

In this section, we analyze the validity of the proposed smoothed regression bootstrap for the inference of the boundary coefficient $\boldsymbol{\gamma}_0$ and the regression coefficient $\boldsymbol{\beta}_0$. Our proofs include two parts. In Section D.1, we presents some conditions for a general bootstrap population, under which the consistency of the bootstrap is shown. In Section D.2, we verify the proposed smoothed regression bootstrap satisfies these conditions, and hence establish its consistency.

D.1. Sufficient conditions for a consistent bootstrap for the segmented regressions. Given a sample $\mathcal{D}_T$ from the model of segmented regression (2.1) of the main paper, suppose the LSE for $\boldsymbol{\beta}_0$ obtained with $\mathcal{D}_T$ is $\widehat{\boldsymbol{\beta}} = (\widehat{\boldsymbol{\beta}}_1^{\mathrm{T}}, \cdots, \widehat{\boldsymbol{\beta}}_4^{\mathrm{T}})^{\mathrm{T}}$, and the centriod of the LSEs for $\boldsymbol{\gamma}_0$ is $\widehat{\boldsymbol{\gamma}}^c$. To simplify notations, in this section we use $\widehat{\boldsymbol{\gamma}}$ for $\widehat{\boldsymbol{\gamma}}^c$. Let $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\beta}})$. The model to generate the bootstrap resamples is

$$Y = \sum_{k=1}^{4} \boldsymbol{X}^{\mathrm{T}}\widehat{\boldsymbol{\beta}}_k \mathbb{1}\{\boldsymbol{Z} \in R_k(\widehat{\boldsymbol{\gamma}}^c)\} + \varepsilon, \tag{D.1}$$

where $(\boldsymbol{X}, \boldsymbol{Z}, \varepsilon) \sim \widehat{\mathbb{Q}}_h$, which generate the bootstrap population that mirrors the population distribution $\mathbb{P}_0$ that generates the original sample $\mathcal{D}_T$. Let $\{Y_i^*, \boldsymbol{X}_i^*, \boldsymbol{Z}_i^*\}_{i=1}^{m_T}$ be a bootstrap resample from (D.1), we denote by $\widehat{\mathbb{Q}}_h^*$ as its empirical measure. The LSEs obtained with the bootstrap resample are $\widehat{\boldsymbol{\theta}}^* = (\widehat{\boldsymbol{\gamma}}^*, \widehat{\boldsymbol{\beta}}^*)$ such that

$$\begin{aligned} \widehat{\mathbb{Q}}_h^*\{m(\boldsymbol{W}^*, \widehat{\boldsymbol{\theta}}^*)\} &= \min_{\boldsymbol{\theta}\in\Theta} \widehat{\mathbb{Q}}_h^*\{m(\boldsymbol{W}^*, \boldsymbol{\theta})\} \\ &= \min_{\boldsymbol{\theta}\in\Theta} \frac{1}{m_T}\sum_{i=1}^{m_T}[Y_i^* - \{\sum_{k=1}^{4} \boldsymbol{X}_i^*\boldsymbol{\beta}_k \mathbb{1}(\boldsymbol{Z}_i^* \in R_k(\boldsymbol{\gamma})\}]^2. \end{aligned} \tag{D.2}$$

Let the bootstrap LSE set for $\boldsymbol{\gamma}$ be $\widehat{G}^*$, whose centriod is denoted as $\widehat{\boldsymbol{\gamma}}^{*c}$.

The sufficient conditions for a consistent boostrap for the segmented regressions are listed as follows.

(C1) [Consistency] $\widehat{\boldsymbol{\theta}} \to \boldsymbol{\theta}_0$.

(C2) [Moment conditions] $\limsup_{T\to\infty} \widehat{\mathbb{Q}}_h(\|\boldsymbol{X}\|^4) < \infty$ and $\limsup_{T\to\infty} \widehat{\mathbb{Q}}_h(\varepsilon^4) < \infty$.

(C3) $\widehat{\mathbb{Q}}_h(\varepsilon_Q|\boldsymbol{X},\boldsymbol{Z})=0$ and $\widehat{\mathbb{Q}}_h(\varepsilon_Q^2)\to\mathbb{P}_0(\varepsilon^2)$.

(C4) Suppose that $U(\boldsymbol{X},\boldsymbol{Z})$ is a function of $(\boldsymbol{X},\boldsymbol{Z})$ with $\mathbb{P}_0\{|U(\boldsymbol{X},\boldsymbol{Z})|\}<\infty$, then

$$\sup_{R\subset\mathcal{Z}}\left|\widehat{\mathbb{Q}}_h\{U(\boldsymbol{X},\boldsymbol{Z})\mathbb{1}(\boldsymbol{Z}\in R)\}-\mathbb{P}_0\{U(\boldsymbol{X},\boldsymbol{Z})\mathbb{1}(\boldsymbol{Z}\in R)\}\right|\to 0. \tag{D.3}$$

(C5) There exist some constants δ_1 and $c_1>0$ such that for each $l=1$ and 2 and any $\epsilon\in(0,\delta_1)$, it holds that $\widehat{\mathbb{Q}}_h\{\mathbb{1}(|q_l|<\epsilon)|\boldsymbol{Z}_{-1,l}\}>c_1\epsilon$ almost surely.

(C6) There exists some constant $r>8$ such that for each $l=1$ and 2, there exists a neighborhood $\mathcal{N}_l$ for $\boldsymbol{\gamma}_{l0}$ such that $\sup_{\boldsymbol{\gamma}\in\mathcal{N}_l}\widehat{\mathbb{Q}}_h(\|\boldsymbol{X}\|^r|\boldsymbol{Z}_l^{\mathrm{T}}\boldsymbol{\gamma}=0)<\infty$, $\inf_{\boldsymbol{\gamma}\in\mathcal{N}_l}\widehat{\mathbb{Q}}_h(\|\boldsymbol{X}\||\boldsymbol{Z}_l^{\mathrm{T}}\boldsymbol{\gamma}=0)>0$ and $\sup_{\boldsymbol{\gamma}\in\boldsymbol{\gamma}_l}\widehat{\mathbb{Q}}_h(\varepsilon^r|\boldsymbol{Z}_l^{\mathrm{T}}\boldsymbol{\gamma}=0)<\infty$.

(C7) For each $l\in\{1,2\}$, as $T\to\infty$ the following hold.

(i) Let $\widetilde{f}_{\boldsymbol{Z}_l}$ be the density function of $\boldsymbol{Z}_l$ under $\widehat{\mathbb{Q}}_h$ and $f_{\boldsymbol{Z}_l}$ be the density function of $\boldsymbol{Z}_l$ under $\mathbb{P}_0$, then $\|\widetilde{f}_{\boldsymbol{Z}_l}-f_{\boldsymbol{Z}_l}\|_\infty\to 0$;

(ii) For each $(j,k)\in\mathcal{S}(l)$,

$$\widehat{\mathbb{Q}}_h\left\{e^{it\xi_Q^{(j,k)}}|q_{l,Q}=0,\boldsymbol{Z}_{-1,l}\right\}\to\mathbb{P}_0\left\{e^{it\xi^{(j,k)}}|q_l=0,\boldsymbol{Z}_{-1,l}\right\}\ \text{almost surely.} \tag{D.4}$$

(iii) Under $\widehat{\mathbb{Q}}_h$, the conditional density $\widetilde{f}_{\xi_Q^{(j,k)}|(q_{l,Q},\boldsymbol{Z}_{-1,l})}(\xi|q,\boldsymbol{z})$ and $\widetilde{f}_{q_{l,Q}|\boldsymbol{Z}_{-1,l}}(q|\boldsymbol{z})$ are continuous at $q=0$ and bounded by some $0<F<\infty$ for any $\xi\in\mathbb{R}$ and $\boldsymbol{z}\in\mathcal{Z}_{-1,l}$;

The following Lemmas D.1–D.5 will establish that under Conditions *(C1)–(C7)*, the asymptotic distributions of the bootstrap estimators are the same as that of the estimators obtained with the sample $\mathcal{D}_T$. The proofs essentially mimics that in Section B, while require careful verification for the validity of replacing $(\mathbb{P}_0,\boldsymbol{\theta}_0)$ with its bootstrap counterpart $(\widehat{\mathbb{Q}}_h,\widehat{\boldsymbol{\theta}})$.

LEMMA D.1. *Assume that Assumptions 1-5 and Conditions (C1)–(C4) hold, then* $\widehat{\boldsymbol{\theta}}^*\xrightarrow{P}\boldsymbol{\theta}_0$.

PROOF. First, we show $\sup_{\boldsymbol{\theta}\in\Theta}\left|\widehat{\mathbb{Q}}_h\{m(\boldsymbol{W},\boldsymbol{\theta})\}-\mathbb{P}_0\{m(\boldsymbol{W},\boldsymbol{\theta})\}\right|\to 0$. For any $\boldsymbol{\theta}$, under $\widehat{\mathbb{Q}}_h$ where $Y=\sum_{k=1}^4\boldsymbol{X}^{\mathrm{T}}\widehat{\boldsymbol{\beta}}\mathbb{1}\{\boldsymbol{Z}\in R(\widehat{\gamma})\}+\varepsilon$,

$$\begin{aligned}
&\widehat{\mathbb{Q}}_h\{m(\boldsymbol{W},\boldsymbol{\theta})\}=\widehat{\mathbb{Q}}_h(\varepsilon_Q^2)\\
&+\sum_{k=1}^4\widehat{\mathbb{Q}}_h[\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_k-\widehat{\boldsymbol{\beta}}_k)\}^2\mathbb{1}^{(k)}(\widehat{\boldsymbol{\gamma}})\mathbb{1}^{(k)}(\boldsymbol{\gamma})]+\sum_{k\neq j}\widehat{\mathbb{Q}}_h[\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_j-\widehat{\boldsymbol{\beta}}_k)\}^2\mathbb{1}^{(k)}(\widehat{\boldsymbol{\gamma}})\mathbb{1}^{(j)}(\boldsymbol{\gamma})]\\
&+2\sum_{k=1}^4\widehat{\mathbb{Q}}_h[\varepsilon\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_k-\widehat{\boldsymbol{\beta}}_k)\mathbb{1}^{(k)}(\widehat{\boldsymbol{\gamma}})\mathbb{1}^{(k)}(\boldsymbol{\gamma})]+2\sum_{k\neq j}\widehat{\mathbb{Q}}_h[\varepsilon\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_j-\widehat{\boldsymbol{\beta}}_k)\mathbb{1}^{(k)}(\widehat{\boldsymbol{\gamma}})\mathbb{1}^{(j)}(\boldsymbol{\gamma})]\\
=:&A_Q+B_{1,Q}(\boldsymbol{\theta})+B_{2,Q}(\boldsymbol{\theta})+C_{1,Q}(\boldsymbol{\theta})+C_{2,Q}(\boldsymbol{\theta}),\ \text{say.}
\end{aligned}$$

Similarly, under P_0 where $Y=\sum_{k=1}^4\boldsymbol{X}^{\mathrm{T}}\boldsymbol{\beta}_0\mathbb{1}^{(k)}(\boldsymbol{\gamma}_0)+\varepsilon$,

$$\begin{aligned}
&\mathbb{P}_0\{m(\boldsymbol{W},\boldsymbol{\theta})\}=\mathbb{P}_0(\varepsilon^2)\\
&+\sum_{k=1}^4\mathbb{P}_0[\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_k-\boldsymbol{\beta}_{0,k})\}^2\mathbb{1}^{(k)}(\boldsymbol{\gamma}_0)\mathbb{1}^{(k)}(\boldsymbol{\gamma})]+\sum_{k\neq j}\mathbb{P}_0[\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_j-\boldsymbol{\beta}_{0,k})\}^2\mathbb{1}^{(k)}(\boldsymbol{\gamma}_0)\mathbb{1}^{(j)}(\boldsymbol{\gamma})]
\end{aligned}$$

$$+2\sum_{k=1}^{4}\mathbb{P}_0[\varepsilon \boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_k-\boldsymbol{\beta}_{0,k})\mathbb{1}^{(k)}(\gamma_0)\mathbb{1}^{(k)}(\boldsymbol{\gamma})]+2\sum_{k\neq j}\mathbb{P}_0[\varepsilon \boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_j-\boldsymbol{\beta}_{0,k})\mathbb{1}^{(k)}(\gamma_0)\mathbb{1}^{(j)}(\boldsymbol{\gamma})]$$

$=:A_P+B_{1,P}(\boldsymbol{\theta})+B_{2,P}(\boldsymbol{\theta})+C_{1,P}(\boldsymbol{\theta})+C_{2,P}(\boldsymbol{\theta})$, say.

Therefore, it suffices to show that $A_Q\to A_P, \sup_{\boldsymbol{\theta}\in\Theta}|B_{i,Q}(\boldsymbol{\theta})-B_{i,P}(\boldsymbol{\theta})|\to 0$ and $\sup_{\boldsymbol{\theta}\in\Theta}|C_{i,Q}(\boldsymbol{\theta})-C_{i,P}(\boldsymbol{\theta})|\to 0$ for $i=1,2$. The first part $A_Q\to A_P$ is because of Condition *(C3)*. Denote $B_{1,Q}(\boldsymbol{\theta})=\sum_{k=1}^4 B_{1,k,Q}$ and $B_{1,P}(\boldsymbol{\theta})=\sum_{k=1}^4 B_{1,k,P}$. Then for each k, using the triangle inequality, we obtain

$$|B_{1,k,Q}(\boldsymbol{\theta})-B_{1,k,P}(\boldsymbol{\theta})|\le D_1(\boldsymbol{\theta})+D_2(\boldsymbol{\theta})+D_3(\boldsymbol{\theta}), \tag{D.5}$$

where

$$D_1(\boldsymbol{\theta})=\left|\widehat{\mathbb{Q}}_h[\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_k-\widehat{\boldsymbol{\beta}}_k)\}^2\mathbb{1}^{(k)}(\widehat{\boldsymbol{\gamma}})\mathbb{1}^{(k)}(\boldsymbol{\gamma})]-\widehat{\mathbb{Q}}_h[\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_k-\boldsymbol{\beta}_{0,k})\}^2\mathbb{1}^{(k)}(\widehat{\boldsymbol{\gamma}})\mathbb{1}^{(k)}(\boldsymbol{\gamma})]\right|,$$

$$D_2(\boldsymbol{\theta})=\left|\widehat{\mathbb{Q}}_h[\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_k-\boldsymbol{\beta}_{0,k})\}^2\mathbb{1}^{(k)}(\widehat{\boldsymbol{\gamma}})\mathbb{1}^{(k)}(\boldsymbol{\gamma})]-\widehat{\mathbb{Q}}_h[\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_k-\boldsymbol{\beta}_{0,k})\}^2\mathbb{1}^{(k)}(\gamma_0)\mathbb{1}^{(k)}(\boldsymbol{\gamma})]\right|,$$

$$D_3(\boldsymbol{\theta})=\left|\widehat{\mathbb{Q}}_h[\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_k-\boldsymbol{\beta}_{0,k})\}^2\mathbb{1}^{(k)}(\gamma_0)\mathbb{1}^{(k)}(\boldsymbol{\gamma})]-\mathbb{P}_0[\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_k-\boldsymbol{\beta}_{0,k})\}^2\mathbb{1}^{(k)}(\gamma_0)\mathbb{1}^{(k)}(\boldsymbol{\gamma})]\right|.$$

For $D_1(\boldsymbol{\theta})$, it can be bounded by

$$\begin{aligned}\sup_{\boldsymbol{\theta}\in\Theta}D_1(\boldsymbol{\theta})\le&\sup_{\boldsymbol{\theta}\in\Theta}\widehat{\mathbb{Q}}_h\left|\left\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_k-\widehat{\boldsymbol{\beta}}_k)\right\}^2-\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_k-\boldsymbol{\beta}_{0,k})\}^2\right|\\ =&\sup_{\boldsymbol{\theta}\in\Theta}\widehat{\mathbb{Q}}_h\left|\left\{\boldsymbol{X}^{\mathrm{T}}\left(2\boldsymbol{\beta}_k-\widehat{\boldsymbol{\beta}}_k-\boldsymbol{\beta}_{0,k}\right)\right\}\left\{\boldsymbol{X}^{\mathrm{T}}\left(\boldsymbol{\beta}_{0,k}-\widehat{\boldsymbol{\beta}}_k\right)\right\}\right|\\ \le&\sqrt{\widehat{\mathbb{Q}}_h\left\{\boldsymbol{X}^{\mathrm{T}}\left(\boldsymbol{\beta}_{0,k}-\widehat{\boldsymbol{\beta}}_k\right)\right\}^2}\sup_{\boldsymbol{\theta}\in\Theta}\sqrt{\widehat{\mathbb{Q}}_h\left\{\boldsymbol{X}^{\mathrm{T}}\left(2\boldsymbol{\beta}_k-\widehat{\boldsymbol{\beta}}_k-\boldsymbol{\beta}_{0,k}\right)\right\}^2},\end{aligned} \tag{D.6}$$

where (D.6) converges to 0 is because its first term converges to 0 by $\boldsymbol{\beta}_{0,k}\to\widehat{\boldsymbol{\beta}}_k$ and Cauchy-Schwartz inequality, and its second term is uniformly bounded since $\limsup_T\widehat{\mathbb{Q}}_h\{\|\boldsymbol{X}\|^4\}<\infty$ and Θ is compact. For $D_2(\boldsymbol{\theta})$,

$$\begin{aligned}D_2(\boldsymbol{\theta})\le&\widehat{\mathbb{Q}}_h\left|\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_k-\boldsymbol{\beta}_{0,k})\}^2\mathbb{1}^{(k)}(\boldsymbol{\gamma})\left\{\mathbb{1}^{(k)}(\widehat{\boldsymbol{\gamma}})-\mathbb{1}^{(k)}(\gamma_0)\right\}\right|\\ &\sqrt{\widehat{\mathbb{Q}}_h\left[\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_k-\boldsymbol{\beta}_{0,k})\}^4\right]}\sqrt{\widehat{\mathbb{Q}}_h\left\{\left|\mathbb{1}^{(k)}(\widehat{\boldsymbol{\gamma}})-\mathbb{1}^{(k)}(\gamma_0)\right|\right\}},\end{aligned} \tag{D.7}$$

where the first term on the right-hand side is uniformly bounded and the second term converges to zero by the dominated convergence theorem and $\widehat{\boldsymbol{\gamma}}\to\gamma_0$ in *(C1)*, we have $\sup_{\boldsymbol{\theta}\in\Theta}D_2(\boldsymbol{\theta})\to 0$. For $D_3(\boldsymbol{\theta})$, let δ_i be the i-th element of $\boldsymbol{\beta}_k-\boldsymbol{\beta}_{0,k}$, then

$$D_3(\boldsymbol{\theta})\le\sum_{i,j\in[p]}\delta_i\delta_j\left|\widehat{\mathbb{Q}}_h\left\{X_iX_j\mathbb{1}^{(k)}(\gamma_0)\mathbb{1}^{(k)}(\boldsymbol{\gamma})\right\}-\mathbb{P}_0\left\{X_iX_j\mathbb{1}^{(k)}(\gamma_0)\mathbb{1}^{(k)}(\boldsymbol{\gamma})\right\}\right|.$$

By the compactness of Θ, $\delta_i\delta_j$ is uniformly bounded. Then from (D.3) in *(C4)*, we obtain $\sup_{\boldsymbol{\theta}\in\Theta}D_3(\boldsymbol{\theta})\to 0$. By (D.5) and triangle inequality, $\sup_{\boldsymbol{\theta}\in\Theta}|B_{1,k,Q}(\boldsymbol{\theta})-B_{1,k,P}(\boldsymbol{\theta})|\to 0$. Summing accross k results in $\sup_{\boldsymbol{\theta}\in\Theta}|B_{1,Q}(\boldsymbol{\theta})-B_{1,P}(\boldsymbol{\theta})|\to 0$.

With the same argument as above except for replacing $\boldsymbol{\beta}_k$ by $\boldsymbol{\beta}_j$ and $\mathbb{1}^{(k)}(\boldsymbol{\gamma})$ by $\mathbb{1}^{(j)}(\boldsymbol{\gamma})$, we can show $\sup_{\boldsymbol{\theta}\in\Theta}|B_{2,Q}(\boldsymbol{\theta})-B_{2,P}(\boldsymbol{\theta})|\to 0$. Similarly, using the above decomposition argument and with Conditions *(C2)*, *(C4)* and *(C1)*, it can be readily shown that $\sup_{\boldsymbol{\theta}\in\Theta}|C_{i,Q}(\boldsymbol{\theta})-C_{i,P}(\boldsymbol{\theta})|\to 0$ for $i=1,2$. Combining the above pieces gives

$$\sup_{\boldsymbol{\theta}\in\Theta}\left|\widehat{\mathbb{Q}}_h\{m(\boldsymbol{W},\boldsymbol{\theta})\}-\mathbb{P}_0\{m(\boldsymbol{W},\boldsymbol{\theta})\}\right|\to 0. \tag{D.8}$$

Because (i) $\widehat{\mathbb{Q}}_h^*$ is the empirical measure of $\widehat{\mathbb{Q}}_h$, (ii) $\widehat{\mathbb{Q}}_h\{\sup_{\boldsymbol{\theta}\in\Theta} m(\boldsymbol{W},\boldsymbol{\theta})\} < \infty$ by the condition *(C2)* and the compactness of Θ, and (iii) $\mathcal{F} = \{m(\mathbf{w},\boldsymbol{\theta}), \boldsymbol{\theta}\in\Theta\}$ has a finite VC-dimension, the Glivenko-Cantelli theorem implies that

$$\sup_{\boldsymbol{\theta}\in\Theta}\left|\widehat{\mathbb{Q}}_h^*\{m(\boldsymbol{W}^*,\boldsymbol{\theta})\} - \widehat{\mathbb{Q}}_h\{m(\boldsymbol{W},\boldsymbol{\theta})\}\right| \xrightarrow{P} 0. \tag{D.9}$$

Consequently, from (D.8) and (D.9) we have

$$\sup_{\boldsymbol{\theta}\in\Theta}\left|\widehat{\mathbb{Q}}_h^*\{m(\boldsymbol{W}^*,\boldsymbol{\theta})\} - \mathbb{P}_0\{m(\boldsymbol{W},\boldsymbol{\theta})\}\right| \xrightarrow{P} 0. \tag{D.10}$$

Because of (D.10) together with the facts that $\boldsymbol{\theta}\mapsto\mathbb{P}_0\{m(\boldsymbol{W},\boldsymbol{\theta})\}$ is continuous and $\boldsymbol{\theta}_0$ is the unique minimizer of $\mathbb{P}_0\{m(\boldsymbol{W},\boldsymbol{\theta})\}$ as established in Appendix B, it follows that $\widehat{\boldsymbol{\theta}}^* \xrightarrow{P} \boldsymbol{\theta}_0$ using the similar arguments as in Section B.2. □

LEMMA D.2. *Assume that Assumptions 1-5 and Conditions (C1)–(C6) hold. Then* $\sqrt{m_T}(\widehat{\boldsymbol{\beta}}^* - \widehat{\boldsymbol{\beta}}) = O_p(1)$ *and* $m_T(\widehat{\boldsymbol{\gamma}}^* - \boldsymbol{\gamma}) = O_p(1)$.

Proof. From Lemma D.1 we know that $\widehat{\boldsymbol{\beta}}^* - \widehat{\boldsymbol{\beta}} = o_p(1)$ and $\widehat{\boldsymbol{\gamma}}^* - \boldsymbol{\gamma} = o_p(1)$. The proof of the convergence rate of $\widehat{\boldsymbol{\beta}}^*$ and $\widehat{\boldsymbol{\gamma}}^*$ is analogous to the proof of $\widehat{\boldsymbol{\beta}}$ and $\widehat{\boldsymbol{\gamma}}$ in Appendix B.

First, because of the conditional zero mean condition of ε in *(C3)*, we can decompose $\widehat{\mathbb{Q}}_h\{m(\boldsymbol{W},\boldsymbol{\theta}) - m(\boldsymbol{W},\boldsymbol{\theta}_Q)\}$ as

$$\begin{aligned}\widehat{\mathbb{Q}}_h\{m(\boldsymbol{W},\boldsymbol{\theta}) - m(\boldsymbol{W},\boldsymbol{\theta}_Q)\} &= \sum_{j=1}^{4}\widehat{\mathbb{Q}}_h[\{\boldsymbol{X}^{\mathrm{T}}(\widehat{\boldsymbol{\beta}}_j - \boldsymbol{\beta}_j)\}^2\mathbb{1}^{(j)}(\widehat{\boldsymbol{\gamma}})\mathbb{1}^{(j)}(\boldsymbol{\gamma})] \\ &\quad + \sum_{i=1}^{4}\sum_{k\neq i}^{4}\widehat{\mathbb{Q}}_h[(\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_{Q,i} - \boldsymbol{\beta}_k))^2\mathbb{1}^{(i)}(\widehat{\boldsymbol{\gamma}})\mathbb{1}^{(k)}(\boldsymbol{\gamma})] \\ &=: \sum_{j=1}^{4} J_j^Q(\boldsymbol{\theta}) + \sum_{i=1}^{4}\sum_{k\neq i}^{4} G_{ik}^Q(\boldsymbol{\theta}), \quad \text{say.}\end{aligned} \tag{D.11}$$

Because $\widehat{\boldsymbol{\gamma}} \to \boldsymbol{\gamma}_0$ and (D.3), it can be shown that

$$\sup_{i,j\in[p]}\sup_{\boldsymbol{\gamma}\in\Gamma}\left|\widehat{\mathbb{Q}}_h\left\{X_iX_j\mathbb{1}^{(j)}(\widehat{\boldsymbol{\gamma}})\mathbb{1}^{(j)}(\boldsymbol{\gamma})\right\} - \mathbb{P}_0\left\{X_iX_j\mathbb{1}^{(j)}(\boldsymbol{\gamma}_0)\mathbb{1}^{(j)}(\boldsymbol{\gamma})\right\}\right| \to 0, \tag{D.12}$$

following similar arguments as for $D_2(\boldsymbol{\theta})$ and $D_3(\boldsymbol{\theta})$ in the previous lemma. Since the smallest eigenvalue of $\mathbb{P}_0\left\{\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\mathbb{1}^{(j)}(\boldsymbol{\gamma}_0)\mathbb{1}^{(j)}(\boldsymbol{\gamma})\right\}$ is uniformly bounded away from 0, (D.12) implies that the smallest eigenvalue of $\widehat{\mathbb{Q}}_h\left\{\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\mathbb{1}^{(j)}(\widehat{\boldsymbol{\gamma}})\mathbb{1}^{(j)}(\boldsymbol{\gamma})\right\}$ is uniformly bounded away from 0 if $\boldsymbol{\gamma}$ is in some neighborhood of $\boldsymbol{\gamma}_0$, for $T \geq T_0$ with some $T_0 > 0$, because the entry-wise convergence of matrices can imply the convergence of eigenvalues, which can be easily seen from the perspective of characteristic polynomials. This implies that

$$J_j^Q(\boldsymbol{\theta}) \geq \|\widehat{\boldsymbol{\beta}}_j - \boldsymbol{\beta}_j\|^2,$$

for $j \in \{1,\cdots,4\}$ and $T \geq T_0$.

With Conditions *(C4)* and *(C5)*, which imply that Assumptions 3.(ii), 4.(i) and 4.(iii) hold when replacing $\mathbb{P}$ with $\widehat{\mathbb{Q}}_h$, the moment inequalies in Lemma A.2 hold under the bootstrap population $\widehat{\mathbb{Q}}_h$. Then, with the same argument as in Step 1 of the proof of Theorem 3.1, it can be shown that for any $\boldsymbol{\gamma}$ in some neighborhood of $\widehat{\boldsymbol{\gamma}}$,

$$\frac{J_{k_l}^Q(\boldsymbol{\theta}) + J_{i_l}^Q(\boldsymbol{\theta})}{2} + G_{i_lk_l}^Q(\boldsymbol{\theta}) + G_{k_li_l}^Q(\boldsymbol{\theta}) \gtrsim \left(\|\boldsymbol{\beta}_{Q,i_l} - \boldsymbol{\beta}_{i_l}\|^2 + \|\boldsymbol{\beta}_{Q,k_l} - \boldsymbol{\beta}_{k_l}\|^2 + \|\widehat{\boldsymbol{\gamma}}_l - \boldsymbol{\gamma}_l\|\right),$$

where k_l and i_l are defined the same as in Appendix B, which further implies

$$\widehat{\mathbb{Q}}_h \{m(\boldsymbol{W}, \boldsymbol{\theta}) - m(\boldsymbol{W}, \boldsymbol{\theta}_Q)\} \gtrsim \|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\|^2 + \|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\|, \tag{D.13}$$

for $\boldsymbol{\gamma}$ in some neighborhood of $\widehat{\boldsymbol{\gamma}}$.

Let $\mathbb{G}_T^* = \sqrt{m_T}(\widehat{\mathbb{Q}}_h^* - \widehat{\mathbb{Q}}_h)$. By inspecting the proofs of Lemmas A.4–A.6, we notice that these lemmas can be established once we have the moment inequalities outlined in Lemma A.3, whose conditions hold if we replace the population $\mathbb{P}_0$ by $\widehat{\mathbb{Q}}_h^*$. Therefore, we can replace $\mathbb{G}_T$ in Lemma A.6 by $\mathbb{G}_T^*$, under Conditions *(C4)–(C6)*. Then, with the same arguments as in Step 2 in Section B.4, we obtain

$$\|\widehat{\boldsymbol{\beta}}^* - \boldsymbol{\beta}\|^2 + \|\widehat{\boldsymbol{\gamma}}^* - \boldsymbol{\gamma}\| \lesssim \|\widehat{\boldsymbol{\beta}}^* - \widehat{\boldsymbol{\beta}}\|^2 o_p(1) + 4\lambda\|\widehat{\boldsymbol{\gamma}}^* - \widehat{\boldsymbol{\gamma}}\| + O_p(m_T^{-1}),$$

for any $\lambda \in (0,1)$, which implies that $\|\widehat{\boldsymbol{\beta}}^* - \widehat{\boldsymbol{\beta}}\|^2 = O_p(m_T^{-1})$, and thus $\|\widehat{\boldsymbol{\gamma}}^* - \widehat{\boldsymbol{\gamma}}\| = O_p(m_T^{-1})$. □

We now proceed to derive a result similar to Lemma B.1.

LEMMA D.3. *Assume that Assumptions 1-5 and Conditions (C1)–(C6) hold. Then uniformly for* $\boldsymbol{h} = (\boldsymbol{u}^\top, \boldsymbol{v}^\top)^\top$ *in any compact set in* $\mathbb{R}^{4p+d_1+d_2}$,

$$\begin{aligned} & m_T \widehat{\mathbb{Q}}_h^* \left\{ m(\boldsymbol{W}^*, \widehat{\boldsymbol{\beta}} + \frac{\boldsymbol{u}}{\sqrt{m_T}}, \widehat{\boldsymbol{\gamma}} + \frac{\boldsymbol{v}}{m_T}) - m(\boldsymbol{W}, \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\gamma}}) \right\} \\ =& D_T^*(\boldsymbol{v}) - 2W_T^*(\boldsymbol{u}) + o_P(1), \end{aligned} \tag{D.14}$$

where

$$W_T^*(\boldsymbol{u}) = \sum_{j=1}^{4} \left[\sqrt{m_T} \widehat{\mathbb{Q}}_h^* \left\{ \boldsymbol{u}_j^\top \boldsymbol{X} \varepsilon_Q \mathbb{1}^{(j)}(\widehat{\boldsymbol{\gamma}}) \right\} + \boldsymbol{u}_j^\top \widehat{\mathbb{Q}}_h \left\{ \boldsymbol{X}\boldsymbol{X}^\top \mathbb{1}^{(j)}(\widehat{\boldsymbol{\gamma}}) \right\} \boldsymbol{u}_j \right],$$

and

$$D_T^*(\boldsymbol{v}) = \sum_{l=1}^{2} \sum_{(j,k)\in\mathcal{S}(l)} m_T \widehat{\mathbb{Q}}_h^* \left[\xi_Q^{(j,k)} \mathbb{1} \left\{ s_l^{(j)} \left(m_T q_{l,Q} + \boldsymbol{Z}_{-1,l}^\top \boldsymbol{v}_{-1,l} \right) \le 0 < s_l^{(j)} m_T q_{l,Q} \right\} \right],$$

with $\xi_Q^{(j,k)} = \left(\widehat{\boldsymbol{\delta}}_{jk}^\top \boldsymbol{X}\boldsymbol{X}^\top \widehat{\boldsymbol{\delta}}_{jk} + 2\boldsymbol{X}^\top \widehat{\boldsymbol{\delta}}_{jk} \varepsilon_Q \right) \left\{ \mathbb{1}^{(j)}(\widehat{\boldsymbol{\gamma}}) + \mathbb{1}^{(k)}(\widehat{\boldsymbol{\gamma}}) \right\}$,

where $\widehat{\boldsymbol{\delta}}_{jk} = \widehat{\boldsymbol{\beta}}_j - \widehat{\boldsymbol{\beta}}_k$, $q_{l,Q} = \boldsymbol{Z}_l^\top \widehat{\boldsymbol{\gamma}}_l$, $\mathcal{S}(l)$ *is the set of indices of adjacent regions split by the* l*-th hyperplane as defined in (3), and* $s_l^{(j)} = \operatorname{sign}(\boldsymbol{z}^\top \boldsymbol{\gamma}_{l0})$ *for* $\boldsymbol{z} \in D^{(j)}(\boldsymbol{\gamma}_0)$ *as defined in (2).*

Proof. The left-hand side of (D.14) can be decomposed in the same way as (B.32) in the proof of Lemma B.1. It is noted that Lemma B.1 is established by showing the decomposed terms in (B.32) besides $D_T(\boldsymbol{v})$ and $W_T(\boldsymbol{u})$ all converge to 0 in probability with the application of Lemma A.5. With Conditions *(C4)–(C6)*, Lemma A.4 holds with $\mathbb{G}_T$ replaced by $\mathbb{G}_T^*$. It can be derived with similar lines of the proof of Lemma A.5 that

$$\begin{aligned} & \sup_{\|\boldsymbol{\gamma}_l - \boldsymbol{\gamma}_{Q,l}\| \le m_T^{-1}} \sqrt{m_T} \widehat{\mathbb{Q}}_h^* \left\{ U \left| \mathbb{1}_j(\boldsymbol{\gamma}_j) - \mathbb{1}_j(\boldsymbol{\gamma}_{Q,j}) \right| \left| \mathbb{1}_l(\boldsymbol{\gamma}_l) - \mathbb{1}_l(\widehat{\boldsymbol{\gamma}}_l) \right| \right\} = o_p(1), \\ & \sup_{\substack{\|\boldsymbol{\gamma}_j - \boldsymbol{\gamma}_{Q,j}\| \le m_T^{-1} \\ \|\boldsymbol{\gamma}_l - \boldsymbol{\gamma}_{Q,l}\| \le m_T^{-1}}} m_T \widehat{\mathbb{Q}}_h^* \left\{ U \left| \mathbb{1}_j(\boldsymbol{\gamma}_j) - \mathbb{1}_j(\boldsymbol{\gamma}_{Q,j}) \right| \left| \mathbb{1}_l(\boldsymbol{\gamma}_l) - \mathbb{1}_l(\widehat{\boldsymbol{\gamma}}_l) \right| \right\} = o_p(1), \end{aligned} \tag{D.15}$$

for $U = \|\boldsymbol{X}\|^2$ and $U = |\varepsilon_Q|\|\boldsymbol{X}\|$. Then, the above lemma can be proved by following the same arguments as in Section B.6. □

LEMMA D.4. *Assume that Conditions (C1)–(C7) hold. Then the finite-dimensional weak limit of $D_T^*(\boldsymbol{v})$ is the same as $D(\boldsymbol{v})$ as presented in Lemma B.2.*

Proof. The derivation of the finite-dimensional weak limit of $D_T^*(\boldsymbol{v})$ is in parallel to that of $D_T(\boldsymbol{v})$ in the proof of Lemma B.2.

First, as (B.48) in Part 1, $D_T^*(\boldsymbol{v})$ can be expressed as a sum of functionals of some empirical point processes. For each $l \in \{1,2\}$ and $(j,k) \in \mathcal{S}(l)$, we define an empirical point process $\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)}(\cdot) \in M_p(E_l)$, where $E_l = \mathbb{R}_{s_l^{(j)}} \times \mathcal{Z}_{-1,l} \times \mathbb{R}$ as:

$$\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)}(F) := m_T \widehat{\mathbb{Q}}_h^* \left[\mathbb{1}\left\{ (m_T q_{l,Q}, \boldsymbol{Z}_{-1,l}, \xi_Q^{(j,k)}) \in F \right\} \right], \tag{D.16}$$

for each $F = (F_1, F_2, F_3) \in E_l$. Then $D_T^*(\boldsymbol{v})$ can be expressed as

$$D_T^*(\boldsymbol{v}) = \sum_{l=1}^{2} \sum_{(j,k)\in\mathcal{S}(l)} \mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)} \left(\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)} \right). \tag{D.17}$$

where the functional $\mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)}$ is defined in (B.47).

Second, we derive the weak limit of $\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)}$ as in Part 2 of the proof of Lemma B.2. The two ingredients are the calculation of the limit of $\widehat{\mathbb{Q}}_h \left\{ \widehat{\mathbf{N}}_{Q,l,T}^{(j,k)} \right\}$, which is required in Kallenberg's theorem, and the application of Meyer's theorem. First, for any $F = (F_1, F_2, F_3) \in \mathcal{E}_l$, the basis of relatively compact open set in E_l, we claim that:

$$\lim_{T, m_T \to \infty} \widehat{\mathbb{Q}}_h \left\{ \widehat{\mathbf{N}}_{Q,l,T}^{(j,k)} \right\} = \mu_l^{(j,k)}(F), \tag{D.18}$$

where the mean measure $\mu_l^{(j,k)}$ is defined in (B.50). This can be shown as below. Note that

$$\begin{aligned}
\widehat{\mathbb{Q}}_h \left\{ \widehat{\mathbf{N}}_{Q,l,T}^{(j,k)} \right\} =& m_T \widehat{\mathbb{Q}}_h \left[\mathbb{1}\left\{ (m_T q_{l,Q}, \boldsymbol{Z}_{-1,l}, \xi_Q^{(j,k)}) \in F \right\} \right] \\
=& m_T \int_{m_T q \in F_1, \boldsymbol{z}\in F_2, \xi \in F_3} \widetilde{f}_Q^{(i,j)}(q, \boldsymbol{z}, \xi) dq d\boldsymbol{z} d\xi \\
=& \int_{\tilde{q}\in F_1, \boldsymbol{z}\in F_2, \xi\in F_3} \widetilde{f}_Q^{(i,j)} \left(\frac{\tilde{q}}{m_T}, \boldsymbol{z}, \xi \right) d\tilde{q} d\boldsymbol{z} d\xi,
\end{aligned}$$

where $\widetilde{f}_Q^{(i,j)}(q, \boldsymbol{z}, \xi)$ is the joint density function of $(q_{l,Q}, \boldsymbol{Z}_{-1,l}, \xi_Q^{(i,j)})$ under $\widehat{\mathbb{Q}}_h$. The claim (D.18) can be verified as follows.

$$\begin{aligned}
& \int_{\tilde{q}\in F_1, \boldsymbol{z}\in F_2, \xi\in F_3} \widetilde{f}_Q^{(i,j)} \left(\frac{\tilde{q}}{m_T}, \boldsymbol{z}, \xi \right) d\tilde{q} d\boldsymbol{z} d\xi \\
=& \int_{\tilde{q}\in F_1, \boldsymbol{z}\in F_2, \xi\in F_3} \widetilde{f}_{\xi_Q^{(j,k)}|(q_{l,Q}, \boldsymbol{Z}_{-1,l})} \left(\xi | \frac{\tilde{q}}{m_T}, \boldsymbol{z} \right) \widetilde{f}_{q_{l,Q}|\boldsymbol{Z}_{-1,l}} \left(\frac{\tilde{q}}{m_T} | \boldsymbol{z} \right) \widetilde{f}_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z}) d\tilde{q} d\boldsymbol{z} d\xi \\
\xrightarrow{(i)} & \int_{\tilde{q}\in F_1, \boldsymbol{z}\in F_2, \xi\in F_3} \widetilde{f}_{\xi_Q^{(j,k)}|(q_{l,Q}, \boldsymbol{Z}_{-1,l})} (\xi | 0, \boldsymbol{z}) \widetilde{f}_{q_{l,Q}|\boldsymbol{Z}_{-1,l}} (0|\boldsymbol{z}) \widetilde{f}_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z}) d\tilde{q} d\boldsymbol{z} d\xi \quad (\text{as } m_T \to \infty) \\
=& \int_{\tilde{q}\in F_1, \boldsymbol{z}\in F_2} \widehat{\mathbb{Q}}_h \left\{ \xi_Q^{(j,k)} \in F_3 | q_{l,Q} = 0, \boldsymbol{Z}_{-1,l} = \boldsymbol{z} \right\} \widetilde{f}_{q_{l,Q}|\boldsymbol{Z}_{-1,l}} (0|\boldsymbol{z}) \widetilde{f}_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z}) d\tilde{q} d\boldsymbol{z}
\end{aligned}$$

$$\overset{(ii)}{\to} \int_{\tilde{q}\in F_1, \boldsymbol{z}\in F_2} \mathbb{P}_0\left\{\xi^{(j,k)} \in F_3 | q_l = 0, \boldsymbol{Z}_{-1,l} = \boldsymbol{z}\right\} f_{q_l|\boldsymbol{Z}_{-1,l}}(0|\boldsymbol{z})\, f_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z}) d\tilde{q} d\boldsymbol{z} \quad (\text{as } T\to\infty)$$

$$= \int_{\tilde{q}\in F_1, \boldsymbol{z}\in F_2, \xi\in F_3} f_{\xi^{(i,j)}|(q_l, \boldsymbol{Z}_{-1,l})}(\xi \mid 0, \boldsymbol{z})\, f_{q_l|\boldsymbol{Z}_{-1,l}}(0 \mid \boldsymbol{z})\, f_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z}) \mathrm{d}\tilde{q} \mathrm{d}\boldsymbol{z} \mathrm{d}\xi$$

$$= \mu_l^{(j,k)}(F),$$

where (i) is implied by the dominated convergence theorem because of the continuity and boundness of $\widetilde{f}_{\xi_Q^{(j,k)}|(q_{l,Q}, \boldsymbol{Z}_{-1,l})}(\xi|q,\boldsymbol{z})$ and $\widetilde{f}_{q_{l,Q}|\boldsymbol{Z}_{-1,l}}(q|\boldsymbol{z})$ at $q=0$ as assumed in *(C7)*. For (ii), since the characteristic function of $\xi_Q^{(i,j)}|q_{l,Q}, \boldsymbol{Z}_{-1,l}$ under $\widehat{\mathbb{Q}}_h$ converges to that of $\xi^{(i,j)}|q_l, \boldsymbol{Z}_{-1,l}$ under $\mathbb{P}_0$, then $\widehat{\mathbb{Q}}_h\left\{\xi_Q^{(j,k)} \in F_3 | q_{l,Q} = 0, \boldsymbol{Z}_{-1,l} = \boldsymbol{z}\right\} \to \mathbb{P}_0\left\{\xi^{(j,k)} \in F_3 | q_l = 0, \boldsymbol{Z}_{-1,l} = \boldsymbol{z}\right\}$. In addition, it is easy to see that

$$\sup_{\boldsymbol{z}\in\mathcal{Z}_{-1,l}} \left| \widetilde{f}_{q_{l,Q}|\boldsymbol{Z}_{-1,l}}(0|\boldsymbol{z})\, \widetilde{f}_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z}) - f_{q_l|\boldsymbol{Z}_{-1,l}}(0|\boldsymbol{z})\, f_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z}) \right| \to 0$$

as $T\to\infty$, due to $\|\widetilde{f}_{\boldsymbol{Z}_l} - f_{\boldsymbol{Z}_l}\|_\infty \to 0$ assumed in *(C7)*. Then (ii) follows from the dominated convergence theorem.

Since observations under $\widehat{\mathbb{Q}}_h^*$ are i.i.d., for any F with $\mu_l^{(j,k)}(F) > 0$, Meyer's theorem implies that

$$\lim_{m_T\to\infty} \widehat{\mathbb{Q}}_h\left\{\mathbb{1}\left(\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)} = 0\right)\right\} = e^{-\mu_l^{(j,k)}(F)}. \tag{D.19}$$

For F with $\mu_l^{(j,k)}(F) = 0$, (D.19) also holds, since in such the case (D.18) implies $\widehat{\mathbb{Q}}_h\left\{\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)}(F)\right\} \to 0$ as $T\to\infty$, which further implies that $\widehat{\mathbb{Q}}_h\left\{\mathbb{1}\left(\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)} = 0\right)\right\} = 1 = e^{-\mu_l^{(j,k)}(F)}$. Since $\mu_l^{(j,k)}$ is the mean measure of $\mathbf{N}_l^{(j,k)}$ introduced in Part 2 of the proof of Lemma B.2, with the statements (D.18) and (D.19), Kallenberg's theorem (Lemma A.7) implies that for each $l\in\{1,2\}$ and $(j,k)\in\mathcal{S}(l)$, we have $\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)} \Rightarrow \mathbf{N}_l^{(j,k)}$ in $M_p(E_l)$ as $m_T, T\to\infty$. Therefore, $\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)}$ has the same weak limit as $\widehat{\mathbf{N}}_{l,T}^{(j,k)}$.

As derived in Part 3 of the proof of Lemma B.2, the point process $\mathbf{N}_l^{(j,k)}$ has the representation (B.55). By inspecting Part 4 of the proof of Lemma B.2 which shows the asymptotical independence of $\left(\widehat{\mathbf{N}}_{l,T}^{(j,k)}, l\in\{1,2\}, (j,k)\in\mathcal{S}(l)\right)$, we find that to show the asymptotical independence of $\left(\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)}, l\in\{1,2\}, (j,k)\in\mathcal{S}(l)\right)$, it suffices to show that (B.59) holds if $\mathbb{P}$ is replaced by $\widehat{\mathbb{Q}}_h$, which is indeed true since $\|\tilde{f}_{\boldsymbol{Z}} - f_{\boldsymbol{Z}}\|_\infty \to 0$ and the uniform boundness of $f_{(q_l,q_{l'})|(\boldsymbol{Z}_{-1,l}, \boldsymbol{Z}_{-1,l'})}(q,q')$ at a neighborhood of $(0,0)$ implies the uniform boundness of $\widetilde{f}_{(q_{l,Q},q_{l',Q})|(\boldsymbol{Z}_{-1,l}, \boldsymbol{Z}_{-1,l'})}(q,q')$ at the neighborhood, which ensures (B.59) holds when replacing $\mathbb{P}$ is replaced by $\widehat{\mathbb{Q}}_h$. The rest arguments in Part 3 of the proof of Lemma B.2 obviously hold under $\widehat{\mathbb{Q}}_h$ and $\widehat{\mathbb{Q}}_h^*$, since the observations under $\mathbb{P}_T$ are weakly dependent and the observations under $\widehat{\mathbb{Q}}_h^*$ are i.i.d. Therefore, the asymptotical independence of $\left(\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)}, l\in\{1,2\}, (j,k)\in\mathcal{S}(l)\right)$ can be established.

As for adapting Part 4 of the proof of Lemma B.2, it is sufficient to verify that (I)–(III) therein hold under $\widehat{\mathbb{Q}}_h$. Let

$$R_{Q,T} = \mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)}\left(\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)}\right) \quad \text{and} \quad R_{Q,T,M} = \int_{E_{l,M}} g_l^{(j,k)}(x,\boldsymbol{y},z) d\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)}(x,\boldsymbol{y},z).$$

For (I), the arguments, which is mainly the continuous mapping theorem, for showing $R_{T,M} \Rightarrow R_{0,M}$ also implies $R_{Q,T,M} \Rightarrow R_{0,M}$, since the probability of discontinuities is evaluated under the distribution of the limiting process $\mathbf{N}_l^{(j,k)}$. For (II), with the notations in (II) in Part 4 of the proof of Lemma B.2, we first have

$$\begin{aligned}|R_{Q,T} - R_{Q,T,M}| =& m_T \widehat{\mathbb{Q}}_h^* \left\{ |\xi| \mathbb{1}\left(|\xi| \geq M\right) \mathbb{1}\left(m_T q_{Q,l} + \boldsymbol{Z}_{-1}^{\mathrm{T}} \boldsymbol{v}_{-1,l} \leq 0 < m_T q_{Q,l}\right) \right\} \\ =:& m_T \widehat{\mathbb{Q}}_h^*(G_Q(M)), \text{ say.} \end{aligned} \tag{D.20}$$

With Condition *(C4)* we have $\widehat{\mathbb{Q}}_h \left\{ \left| \xi_Q^{(j,k)} \right|^4 | \boldsymbol{Z}_l^{\mathrm{T}} \boldsymbol{\gamma} = 0 \right\} < C$ for some $C < \infty$ if $\boldsymbol{\gamma}$ is in some neighborhood of $\widehat{\boldsymbol{\gamma}}_l$ and each $l \in \{1,2\}$. As in (B.74) it can be readily shown that $\widehat{\mathbb{Q}}_h$

$$\widehat{\mathbb{Q}}_h \left\{ |\xi| \mathbb{1}\left(|\xi| \geq M\right) \mid \boldsymbol{Z}_l^{\mathrm{T}} \boldsymbol{\gamma} = 0 \right\} = O(M^{-1}). \tag{D.21}$$

Using (D.21) and with the similar arguments as in the proof of Lemma A.4 (i), we can show that which implies $\widehat{\mathbb{Q}}_h \left\{ |G_t(M)| \right\} = O((M m_T)^{-1})$, which implies $\widehat{\mathbb{Q}}_h \left\{ |R_{Q,T} - R_{Q,T,M}| \right\} = O(M^{-1})$ due to (D.21). Then

$$\lim_{M \to \infty} \limsup_{T \to \infty} \widehat{\mathbb{Q}}_h \left\{ |R_{Q,T} - R_{Q,T,M}| > \varepsilon \right\} \to 0,$$

for any $\varepsilon > 0$ according to the Markov inequality, which verifies (II). Since (III) in Part 4 of the proof of Lemma B.2 is for the truncation error of $R_0 = \mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)} \left(\mathbf{N}_{l,T}^{(j,k)} \right)$, which is regardless of $\mathbb{P}_0$ or $\widehat{\mathbb{Q}}_h$, it also holds under the bootstrap scenario. Therefore, with (I)–(III) and by applying Theorem 4.2 of Billingsley (1968), $R_{T,Q} \Rightarrow R_0$ as $m_T, \to \infty$, i.e., $\mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)} \left(\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)} \right) \Rightarrow \mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)} \left(\mathbf{N}_{l,T}^{(j,k)} \right)$. Because it has been shown that $\left(\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)}, l \in \{1,2\}, (j,k) \in \mathcal{S}(l) \right)$ are asymptotically independent, we conclude that

$$D_T^*(\boldsymbol{v}) = \sum_{l=1}^{L} \sum_{(j,k) \in \mathcal{S}(l)} \mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)} \left(\widehat{\mathbf{N}}_{Q,l,T}^{(j,k)} \right) \Rightarrow \sum_{l=1}^{L} \sum_{(j,k) \in \mathcal{S}(l)} \mathcal{T}_{l,\boldsymbol{v}_l}^{(j,k)} \left(\mathbf{N}_{l,T}^{(j,k)} \right), \tag{D.22}$$

as $m_T, T \to \infty$, where the right-hand side of (D.22) is identical to that of (B.77), which is the weak limit of $D_T(\boldsymbol{v})$, the proof is completed. □

Let $\widehat{\boldsymbol{\gamma}}^{*c} = C(\widehat{G}^*)$ be the centriod of the LSEs $\widehat{\boldsymbol{\gamma}}^*$ obtained with the bootstrap resample. Let $\mathcal{L}_T^*$ be the distribution of $\{m_T(\widehat{\boldsymbol{\gamma}}^{*c} - \widehat{\boldsymbol{\gamma}}^c), \sqrt{m_T}(\widehat{\boldsymbol{\beta}}^* - \widehat{\boldsymbol{\beta}})\}$ and $\mathcal{L}_T$ be the distribution of $\{T(\widehat{\boldsymbol{\gamma}}^c - \boldsymbol{\gamma}_0), \sqrt{T}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)\}$. The s.e-l-sc of $\{D_T^*\}$ can be obtained with the same proof as for Lemma B.3. With the same arguments as the proof for Theorem 3.3, we can establish that $\mathcal{L}_T^*$ has the same limiting distribution as that of $\mathcal{L}_T$, which implies the following result.

LEMMA D.5. *Assume that Conditions (C1)–(C7) hold, then $\rho(\mathcal{L}_T^*, \mathcal{L}_T) \to 0$ as $T, m_T \to \infty$, for any metric ρ that metrizes weak convergence of distributions.*

D.2. Proof of Theorem 5.1.

PROOF. To show the validity of the smoothed regression bootstrap, we just need to verify Conditions *(C1)–(C7)* hold with the probability approaching 1, conditionally on the data $\{\boldsymbol{W}_t = (Y_t, \boldsymbol{X}_t, \boldsymbol{Z}_t)\}_{t=1}^T$, where under the bootstrap distribution $\widehat{\mathbb{Q}}_h$, the bootstrap sample $(\boldsymbol{X}^*, \boldsymbol{Z}^*) \sim \tilde{F}(\mathbf{x}, \boldsymbol{z})$, whose density function is the nonparametric density estimator $\tilde{f}(\mathbf{x}, \boldsymbol{z})$. First, under Assumptions 6.(i) and (iii), we have $\|\tilde{f}(\mathbf{x}, \boldsymbol{z}) - f(\mathbf{x}, \boldsymbol{z})\|_\infty = o_p(1)$, as a standard

result in kernel regression estimation (Györfi et al., 1989 and Hansen, 2008). Conditioning on $(\boldsymbol{X}^*, \boldsymbol{Z}^*)$, the noise $\varepsilon^* \sim \tilde{\sigma}(\boldsymbol{X}^*, \boldsymbol{Z}^*)e^*$, where $e^* \sim \widehat{F}_e$ which is independent of $\tilde{f}(\mathbf{x}, \boldsymbol{z})$. The bootstrap response is generated from

$$Y^* = \sum_{k=1}^{4} (\boldsymbol{X}^*)^{\mathrm{T}} \widehat{\boldsymbol{\beta}}_k \mathbb{1}\left\{\boldsymbol{Z}^* \in R_k(\widehat{\boldsymbol{\gamma}})\right\} + \varepsilon_Q^*. \tag{D.23}$$

Condition *(C1)* is a direct consequence of Theorem 3.1. Let $\tilde{f}(\mathbf{x}) = \int \tilde{f}(\mathbf{x}, \boldsymbol{z}) d\boldsymbol{z}$ and $f(\mathbf{x})$ be the density of $\boldsymbol{X}$ under $\mathbb{P}_0$. Then we have $\|\tilde{f}(\mathbf{x}) - f(\mathbf{x})\|_\infty$ converges to 0 in probability, which is implied by $\|\tilde{f}(\mathbf{x}, \boldsymbol{z}) - f(\mathbf{x}, \boldsymbol{z})\|_\infty \xrightarrow{P} 0$ and the dominidated convergence theorem. Therefore, $\widehat{\mathbb{Q}}_h(\|\boldsymbol{X}\|^4) = \int \|\mathbf{x}\|^4 \tilde{f}(\mathbf{x}) d\mathbf{x}$ converges to $\mathbb{P}_0(\|\boldsymbol{X}\|^4) < \infty$ by the dominidated convergence theorem, which verifies the first condition in *(C2)*. For the second condition of the boundness of $\widehat{\mathbb{Q}}_h(\varepsilon^4)$, we notice that by the independence of $\widehat{F}_e$ and $\tilde{f}(\mathbf{x}, \boldsymbol{z})$,

$$\widehat{\mathbb{Q}}_h(\varepsilon^4) = \int \tilde{\sigma}^4(\boldsymbol{X}, \boldsymbol{Z}) e^4 \tilde{f}(\mathbf{x}, \boldsymbol{z}) d\mathbf{x} d\boldsymbol{z} d\widehat{F}_e(e), \tag{D.24}$$

which is $O_p(1)$ because of (v) of Lemma D.6, the uniform boundness of $\tilde{\sigma}(\mathbf{x}, \boldsymbol{z})$ and $\tilde{f}(\mathbf{x}, \boldsymbol{z})$, which are also compactly supported. Therefore, we conclude that *(C2)* holds in probability approaching 1. Because $\varepsilon = \tilde{\sigma}(\boldsymbol{X}, \boldsymbol{Z})e$, where $e \sim \widehat{F}_e$ is independent of $(\boldsymbol{X}, \boldsymbol{Z})$ and has a zero mean, it holds that $\widehat{\mathbb{Q}}_h(\varepsilon | \boldsymbol{X}, \boldsymbol{Z}) = 0$. As a standard result in local linear regression, Assumptions 6. (i) and (ii) imply $\|\tilde{\sigma}(\mathbf{x}, \boldsymbol{z}) - \sigma(\mathbf{x}, \boldsymbol{z})\|_\infty \xrightarrow{P} 0$, which together with (iv) of Lemma D.6 leads to $\widehat{\mathbb{Q}}_h(\varepsilon_Q^2) \xrightarrow{P} \mathbb{P}_0(\varepsilon^2)$. Therefore, Condition *(C3)* holds in probability. Because $(\boldsymbol{X}, \boldsymbol{Z})$ has a compact support and $\|\tilde{f}(\mathbf{x}, \boldsymbol{z}) - f(\mathbf{x}, \boldsymbol{z})\|_\infty \xrightarrow{P} 0$, applying the dominated convergence therorem yields that (D.3) holds in probability. Therefore, *(C4)* is ensured.

To show *(C5)*, we first note that for any $l \in \{1, 2\}$,

$$\left| \tilde{f}_{q_l, Q | \boldsymbol{Z}_{-1,l}}(q | \boldsymbol{z}) - f_{q_l | \boldsymbol{Z}_{-1,l}}(q | \boldsymbol{z}) \right| = \left| \frac{\tilde{f}_{q_l, Q, \boldsymbol{Z}_{-1,l}}(q, \boldsymbol{z})}{\tilde{f}_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z})} - \frac{f_{q_l, \boldsymbol{Z}_{-1,l}}(q, \boldsymbol{z})}{f_{\boldsymbol{Z}_{-1,l}}(\boldsymbol{z})} \right| \xrightarrow{P} 0, \tag{D.25}$$

for q and $\boldsymbol{z}$ uniformly. Since $f_{q_l | \boldsymbol{Z}_{-1,l}}(q | \boldsymbol{z})$ is bounded for each $\boldsymbol{z} \in \mathcal{Z}_{-1,l}$ and q_l in the neighborhood of 0 as required in Assumption 5. (ii), (D.25) implies that $\tilde{f}_{q_l, Q | \boldsymbol{Z}_{-1,l}}(q | \boldsymbol{z})$ is bounded in probability. Then using the dominated convergence theorem, Condition *(C5)* can be shown. Assumption 6. (i) requires that $\mathcal{X} \times \mathcal{Z}$ is compact and implies that $f_{\boldsymbol{X}|\boldsymbol{Z}}$ is bounded. Hence, for any finite r,

$$\begin{aligned} \widehat{\mathbb{Q}}_h\left(\|\boldsymbol{X}\|^r | \boldsymbol{Z}_l^{\mathrm{T}} \boldsymbol{\gamma} = 0\right) &= \int_{\mathcal{X} \times \mathcal{Z}} \|\mathbf{x}\|^r \mathbb{1}(\boldsymbol{z}^{\mathrm{T}} \boldsymbol{\gamma} = 0) \frac{\tilde{f}_{\boldsymbol{X}, \boldsymbol{Z}_l}(\mathbf{x}, \boldsymbol{z})}{\tilde{f}_{\boldsymbol{Z}_l}(\boldsymbol{z})} d\mathbf{x} d\boldsymbol{z} \\ &\xrightarrow{P} \mathbb{P}_0\left(\|\boldsymbol{X}\|^r | \boldsymbol{Z}_l^{\mathrm{T}} \boldsymbol{\gamma} = 0\right), \end{aligned} \tag{D.26}$$

by the dominated convergence theorem. With the consistency of $\widehat{\boldsymbol{\gamma}}$ and Assumption 4, (D.26) implies the first two conditions in Condition *(C6)*. Since

$$\widehat{\mathbb{Q}}_h\left(\varepsilon^r | \boldsymbol{Z}_l^{\mathrm{T}} \boldsymbol{\gamma} = 0\right) = \int_{\mathbb{R}} x^r d\widehat{F}_e(x) \int_{\mathcal{X} \times \mathcal{Z}} \tilde{\sigma}(\mathbf{x}, \boldsymbol{z}) \mathbb{1}(\boldsymbol{z}^{\mathrm{T}} \boldsymbol{\gamma} = 0) \frac{\tilde{f}_{\boldsymbol{X}, \boldsymbol{Z}_l}(\mathbf{x}, \boldsymbol{z})}{\tilde{f}_{\boldsymbol{Z}_l}(\boldsymbol{z})} d\mathbf{x} d\boldsymbol{z},$$

using Lemma D.6 (v) and Assumption 6. (ii) ensures that $\widehat{\mathbb{Q}}_h\left(\varepsilon^r | \boldsymbol{Z}_l^{\mathrm{T}} \boldsymbol{\gamma} = 0\right) < \infty$ for the r specified in Assumption 4 (iv). Hence, Condition *(C6)* is verified.

For *(C7)*, (i) is a direct consequence of $\|\tilde{f}(\mathbf{x},\boldsymbol{z}) - f(\mathbf{x},\boldsymbol{z})\|_\infty \xrightarrow{P} 0$. For (ii), recall that $\xi_Q^{(j,k)} = \left(\widehat{\boldsymbol{\delta}}_{jk}^{\mathrm{T}} \boldsymbol{X}\boldsymbol{X}^{\mathrm{T}} \widehat{\boldsymbol{\delta}}_{jk} + 2\boldsymbol{X}^{\mathrm{T}} \widehat{\boldsymbol{\delta}}_{jk} \tilde{\sigma}(\boldsymbol{X},\boldsymbol{Z}) e_Q\right) \left\{\mathbb{1}^{(j)}(\widehat{\boldsymbol{\gamma}}) + \mathbb{1}^{(k)}(\widehat{\boldsymbol{\gamma}})\right\}$, to emphasis it is a function of $(\boldsymbol{X},\boldsymbol{Z},e,\widehat{\boldsymbol{\theta}})$, we write $\xi_Q^{(j,k)} = \xi(\boldsymbol{X},\boldsymbol{Z},e,\widehat{\boldsymbol{\theta}})$. Then,

$$\begin{aligned}
\widehat{\mathbb{Q}}_h\left\{e^{it\xi_Q^{(j,k)}} | q_{l,Q}=0, \boldsymbol{Z}_{-1,l}\right\} &= \int e^{it\xi(\mathbf{x},\boldsymbol{z},e,\widehat{\boldsymbol{\theta}})} \mathbb{1}(\boldsymbol{z}^{\mathrm{T}}\widehat{\boldsymbol{\gamma}}_l = 0) \frac{\tilde{f}_{\boldsymbol{X},\boldsymbol{Z}_l}(\mathbf{x},\boldsymbol{z})}{\tilde{f}_{\boldsymbol{Z}_l}(\boldsymbol{z})} d\mathbf{x} d\boldsymbol{z} d\widehat{F}_e(e) \\
&\xrightarrow{P} \int e^{it\xi(\mathbf{x},\boldsymbol{z},e,\boldsymbol{\theta}_0)} \mathbb{1}(\boldsymbol{z}^{\mathrm{T}}\boldsymbol{\gamma}_{l0} = 0) \frac{f_{\boldsymbol{X},\boldsymbol{Z}_l}(\mathbf{x},\boldsymbol{z})}{f_{\boldsymbol{Z}_l}(\boldsymbol{z})} d\mathbf{x} d\boldsymbol{z} dF_e(e) \\
&= \mathbb{P}_0\left\{e^{it\xi^{(j,k)}} | q_l = 0, \boldsymbol{Z}_{-1,l}\right\}, \qquad \text{(D.27)}
\end{aligned}$$

by Lemma D.6 (i) and the dominated convergence theorem. Therefore, *(C7)* (ii) holds in probability. Finally, for *(C7)* (iii) we note that for each $l \in \{1,2\}$,$\boldsymbol{z}_{-1,i} \in \mathcal{Z}_{-1,l}, q \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $|q| < \delta$,

$$\begin{aligned}
&|\tilde{f}_{q_l|\boldsymbol{Z}_{-1,l}}(q|\boldsymbol{Z}_{-1,l}) - \tilde{f}_{q_l|\boldsymbol{Z}_{-1,l}}(0|\boldsymbol{Z}_{-1,l})| \\
\leq &\sum_{i=1}^{2} |\tilde{f}_{q_l|\boldsymbol{Z}_{-1,l}}(q_i|\boldsymbol{Z}_{-1,l}) - f_{q_l|\boldsymbol{Z}_{-1,i}}(q_i|\boldsymbol{Z}_{-1,l})| + |f_{q_l|\boldsymbol{Z}_{-1,l}}(q|\boldsymbol{Z}_{-1,l}) - f_{q_l|\boldsymbol{Z}_{-1,l}}(0|\boldsymbol{Z}_{-1,l})|,
\end{aligned}$$

where $q_1 = q$ and $q_2 = 0$. With (D.25), which shows the first term of the right-hand side of the above inequality is $o_p(1)$, and Assumption 5. (ii), which implies that for any $\varepsilon > 0$, there exists $\delta > 0$ such that the second term is less than ϵ provided that $|q| < \delta$, it can be shown that $\tilde{f}_{q_l|\boldsymbol{Z}_{-1,l}}(q|\boldsymbol{Z}_{-1,l})$ is continuous at 0 for each $\boldsymbol{z}_{-1,l}$ in probability. Similarly, the continuity of $\widetilde{f}_{\xi_Q^{(j,k)}|(q_{l,Q},\boldsymbol{Z}_{-1,l})}(\xi|q,\boldsymbol{z})$ can be shown. Hence, Condition *(C7)* holds with the probability approaching 1. Finally, with Conditions *(C1)*–*(C7)* verified, Theorem 5.1 follows by applying Lemma D.4. □

LEMMA D.6. *Let F_e and φ_e be the distribution function and characteristic function of e, respectively. Then under Assumptions 1-6,*

(i) for any $\eta > 0$, $\sup_{|\xi|\leq\eta}\left\{\left|\int \exp(i\xi x) d\widehat{F}_e - \varphi_e(\xi)\right|\right\} \xrightarrow{P} 0$;

(ii) $\|\widehat{F}_e - F_e\|_\infty \xrightarrow{P} 0$;

(iii) $\int |x| d\widehat{F}_e(x) \xrightarrow{P} \mathbb{P}_0(|e|)$;

(iv) $\int x^2 d\widehat{F}_e(x) \xrightarrow{P} 1$;

(v) $\int x^r d\widehat{F}_e(x) = O_p(1)$, where r is specified in Assumption 4.

PROOF. (i) Let $F_{T,e}$ be the empirical distribution function of $\{e_t\}_{t=1}^T$. Note that

$$\int \exp(i\xi x) d\widehat{F}_e(x) = \exp\left(-it\bar{e}_T\right) \mathbb{P}_T\left\{\exp\left(i\xi\widehat{e}_t\right)\right\}.$$

Hence, for any $|\xi| \leq \eta$ with $\eta > 0$, we have

$$\begin{aligned}
&\left|\int \exp(i\xi x) d\widehat{F}_e - \exp\left(-i\xi\bar{e}_T\right) \int \exp(i\xi x) dF_{T,e}(x)\right| \\
= &\left|\mathbb{P}_T\left\{\exp\left(i\xi\widehat{e}_t\right)\right\} - \mathbb{P}_T\left\{\exp(i\xi e_t)\right\}\right| \leq |\eta| \mathbb{P}_T\left(|\widehat{e}_t - e_t|\right). \qquad \text{(D.28)}
\end{aligned}$$

We claim that

$$\mathbb{P}_T\left(|\widehat{e}_t - e_t|\right) \xrightarrow{P} 0, \qquad \text{(D.29)}$$

which will be shown later. Then (D.28) implies that

$$\sup_{|\xi|\leq\eta}\left\{\left|\int\exp(i\xi x)d\widehat{F}_e-\exp\left(-i\xi\bar{e}_T\right)\int\exp(i\xi x)dF_{T,e}(x)\right|\right\}\xrightarrow{P}0, \tag{D.30}$$

and Lemma D.6 (i) follows from the facts that $\bar{e}_T=\mathbb{P}_T(\widehat{e}_t)\xrightarrow{P}0$, and

$$\sup_{|\xi|\leq\eta}\left|\mathbb{P}_T\left\{\exp(i\xi e_t)\right\}-\mathbb{P}_0\left\{\exp(i\xi e_t)\right\}\right|\xrightarrow{P}0$$

by the ULLN under mixing sequences.

It remains to verify the claim (D.29). For notational simplicity, we denote $\widehat{\sigma}_t:=\tilde{\sigma}(\boldsymbol{X}_t,\boldsymbol{Z}_t)$ and $\sigma_t:=\sigma(\boldsymbol{X}_t,\boldsymbol{Z}_t)$. Then $\sup_{1\leq t\leq T}|\sigma_t-\widehat{\sigma}_t|=o_p(1)$ by Assumption 5.(ii). Note that

$$\begin{aligned}\widehat{e}_t=&\frac{Y_t-\sum_{k=1}^4\boldsymbol{X}_t^{\mathrm{T}}\widehat{\boldsymbol{\beta}}_k\mathbb{1}_t^{(k)}(\widehat{\boldsymbol{\gamma}})}{\widehat{\sigma}_t}\\ =&\frac{\sum_{j=1}^4\boldsymbol{X}_t^{\mathrm{T}}\left(\widehat{\boldsymbol{\beta}}_j-\boldsymbol{\beta}_{j0}\right)\mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}(\widehat{\boldsymbol{\gamma}})}{\widehat{\sigma}_t}+\frac{\sum_{j=1}^4\sum_{i\neq j}^K\boldsymbol{X}_t^{\mathrm{T}}\left(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_{j0}\right)\mathbb{1}_t^{(i)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}(\widehat{\boldsymbol{\gamma}})}{\widehat{\sigma}_t}\\ &+\frac{\sigma_t-\widehat{\sigma}_t}{\widehat{\sigma}_t}e_t+e_t=:E_{1,t}+E_{2,t}+E_{3,t}+e_t,\ \text{ say.}\end{aligned} \tag{D.31}$$

Denote $\hat{E}_{k,T}=\mathbb{P}_T(|E_{k,t}|)$ for $k=1,2,3$. Then to show (D.29), it suffices to show $\hat{E}_{k,T}\xrightarrow{P}0$ as $T\to\infty$. For the first term $E_{1,T}$, we have

$$\begin{aligned}\hat{E}_{1,T}\leq&\sum_{j=1}^4\mathbb{P}_T\left\{\left|\frac{\boldsymbol{X}_t^{\mathrm{T}}\left(\widehat{\boldsymbol{\beta}}_j-\boldsymbol{\beta}_{j0}\right)\mathbb{1}_t^{(j)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}(\widehat{\boldsymbol{\gamma}})}{\widehat{\sigma}_t}\right|\right\}\\ \leq&\sum_{j=1}^4\mathbb{P}_T\left\{\left|\frac{\boldsymbol{X}_t^{\mathrm{T}}\left(\widehat{\boldsymbol{\beta}}_j-\boldsymbol{\beta}_{j0}\right)}{\sigma_t+o_p(1)}\right|\right\}\leq\sum_{j=1}^4\mathbb{P}_T\left\{\left|\frac{\|\boldsymbol{X}_t\|\|\widehat{\boldsymbol{\beta}}_j-\boldsymbol{\beta}_{j0}\|}{\sigma_t+o_p(1)}\right|\right\}\\ =&O_p(T^{-1/2}),\end{aligned} \tag{D.32}$$

since $\sigma_t>\underline{\sigma}>0$ and $\|\widehat{\boldsymbol{\beta}}_j-\boldsymbol{\beta}_{j0}\|=O_p(T^{-1/2})$. For the second term $E_{2,T}$, it is $o_p(1)$ if for each $i\neq j\in\{1,\cdots,4\}$, $\mathbb{P}_T\left\{\left|\boldsymbol{X}_t^{\mathrm{T}}\left(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_{j0}\right)\mathbb{1}_t^{(i)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}(\widehat{\boldsymbol{\gamma}})/\widehat{\sigma}_t\right|\right\}=o_p(1)$, which can be shown as

$$\begin{aligned}&\mathbb{P}_T\left\{\left|\frac{\boldsymbol{X}_t^{\mathrm{T}}\left(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_{j0}\right)\mathbb{1}_t^{(i)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}(\widehat{\boldsymbol{\gamma}})}{\widehat{\sigma}_t}\right|\right\}\\ \leq&\mathbb{P}_T\left\{\left|\frac{\boldsymbol{X}_t^{\mathrm{T}}(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_{i0})}{\widehat{\sigma}_t}\right|\right\}+\mathbb{P}_T\left\{\left|\frac{\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{ij,0}\mathbb{1}_t^{(i)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}(\widehat{\boldsymbol{\gamma}})}{\widehat{\sigma}_t}\right|\right\},\end{aligned}$$

where the first term is $O_p(T^{-1/2})$ from the same reason as for (D.32). For the second term,

$$\mathbb{P}_T\left\{\left|\frac{\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\delta}_{ij,0}\mathbb{1}_t^{(i)}(\boldsymbol{\gamma}_0)\mathbb{1}_t^{(j)}(\widehat{\boldsymbol{\gamma}})}{\widehat{\sigma}_t}\right|\right\}\leq\sum_{l=1}^L\mathbb{P}_T\left\{\frac{\|\boldsymbol{X}_t\|\|\boldsymbol{\delta}_{ij,0}\|}{\sigma_t+o_p(1)}\left|\mathbb{1}_{l,t}(\boldsymbol{\gamma}_{l0})-\mathbb{1}_{l,t}(\widehat{\boldsymbol{\gamma}}_l)\right|\right\},$$

which is $O_p(T^{-1})$ because of (A.20) in Lemma A.4. Therefore, we obtain $\hat{E}_{2,T} = O_p(T^{-1/2})$. For the third term $\hat{E}_{3,T}$, it holds that

$$\hat{E}_{3,T} = \mathbb{P}_T\left(\left|\frac{\sigma_t - \widehat{\sigma}_t}{\widehat{\sigma}_t} e_t\right|\right) \leq \sqrt{\mathbb{P}_T\left(\left|\frac{\sigma_t - \widehat{\sigma}_t}{\widehat{\sigma}_t}\right|^2\right)} \sqrt{\mathbb{P}_T(e_t^2)}.$$

Since $\mathbb{P}_T(e_t^2) = O_p(1)$, and $|\sigma_t - \widehat{\sigma}_t| = o_p(1), \sigma_t < \underline{\sigma}$ uniformly for $t \in \{1, \cdots, T\}$, it yields that $E_{3,T} = o_p(1)$. Combining with (D.31), it yields that for any $t \in \{1, \cdots, T\}$.

$$\mathbb{P}_T(|\widehat{e}_t - e_t|) \leq \hat{E}_{1,T} + \hat{E}_{2,T} + \hat{E}_{3,T} = o_p(1), \tag{D.33}$$

which verifies the claim (D.29), and thus completes the proof for (i).

(ii) By Levy-Cramer continuity theorem, (i) implies that $\widehat{F}_e(x) = F_e(x) + o_p(1)$ for any $x \in \mathbb{R}$. Then (ii) follows from the continuity of F_e and Polya's theorem.

(iii) Note that

$$\begin{aligned}\left|\int |x| d\widehat{F}_e(x) - \mathbb{P}_T(|e_t|)\right| &= |\mathbb{P}_T(|\widehat{e}_t - \bar{e}_T| - |e_t|)| \\ &\leq \mathbb{P}_T(|\widehat{e}_t - e_t|) + |\bar{e}_T| \xrightarrow{P} 0,\end{aligned}$$

implied by (D.29) and $\bar{e}_T = \mathbb{P}_T(\widehat{e}_t) \xrightarrow{P} 0$. Because $\mathbb{P}_T(|e_t|) = \mathbb{P}_0(|e|) + o_p(1)$ by the weak law of large numbers, the conclusion (iii) follows.

(iv) Since $\int x^2 d\widehat{F}_e(x) = \mathbb{P}_T\left(\widehat{e}_t^2\right) - (\bar{e}_T)^2 = \mathbb{P}_T\left(\widehat{e}_t^2\right) + o_p(1)$ and $\mathbb{P}_T(e_t^2) = \mathbb{P}_0(e^2) + o_p(1) = 1 + o_p(1)$, to show (iv) it is sufficient to show that $\mathbb{P}_T\left(\widehat{e}_t^2\right) - \mathbb{P}_T(e_t^2) = o_p(1)$. From (D.31) we have

$$\widehat{e}_t^2 - e_t^2 = (E_{1,t} + E_{2,t} + E_{3,t})^2 + 2(E_{1,t} + E_{2,t} + E_{3,t}) e_t, \tag{D.34}$$

which implies that

$$\begin{aligned}\left|\mathbb{P}_T\left(\widehat{e}_t^2\right) - \mathbb{P}_T(e_t^2)\right| &\leq \mathbb{P}_T\left(\left|\widehat{e}_t^2 - e_t^2\right|\right) \\ &\leq 3\sum_{i=1}^{3} \mathbb{P}_T(E_{i,t}^2) + 2\sqrt{\mathbb{P}_T\{(\sum_{i=1}^{3} E_{i,t})^2\}} \sqrt{\mathbb{P}_T(e_t^2)} \\ &\leq 3\sum_{i=1}^{3} \mathbb{P}_T(E_{i,t}^2) + 2\sqrt{3\sum_{i=1}^{3} \mathbb{P}_T(E_{i,t}^2)} \sqrt{1 + o_p(1)},\end{aligned}$$

by the C_r and Cauchy-Schwartz inequalities. Therefore, $\mathbb{P}_T\left(\widehat{e}_t^2\right) - \mathbb{P}_T(e_t^2) = o_p(1)$ if $\mathbb{P}_T(E_{i,t}^2) = o_p(1)$ for $i = 1, 2, 3$. Since this can be shown in the almost same way as for showing $\mathbb{P}_T(|E_{i,t}|) = o_p(1)$ in the proof of (i), we omit the detailed proof here for simplicity.

(v) Note that

$$\int |x|^r d\widehat{F}_e(x) \leq \sum_{i=0}^{r} \binom{r}{i} |\bar{e}_T|^i \mathbb{P}_T(\widehat{e}_t^{r-i}), \tag{D.35}$$

and $|\bar{e}_T|^i = |\mathbb{P}_T(\widehat{e}_t)|^i = o_p(1)$ for each $1 \leq i \leq r$. Using the expansion (D.31) and the fact that $\mathbb{P}_T(|e_t|^i) = \mathbb{P}_0(|e_t|^i) + o_p(1)$, it is straightforward to show that $\mathbb{P}_T(\widehat{e}_t^i) = O_p(1)$ for each $1 \leq i \leq r$. Therefore, the desired result (v) is verified. □

APPENDIX E: PROOFS FOR SECTION 6

E.1. Proof of Theorem 6.1. In this subsection, we present the proof for Theorem 6.1 of the main paper on the convergence of the four-regime based LS estimator under the segmented models with less than four regimes.

PROOF. Suppose that the true model is

$$Y = \sum_{k=1}^{K_0} \boldsymbol{X}^{\mathrm{T}} \boldsymbol{\beta}_{k0} \mathbb{1}\{\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0)\} + \varepsilon, \tag{E.1}$$

where the number of regimes $K_0 \leq 4$ and the number of splitting hyperplanes $L_0 \leq 2$. In particular, $R_k(\boldsymbol{\gamma}_0) = \mathcal{Z}_1 \times \mathcal{Z}_2$ for the global linear model ($K_0 = 1$), the splitting coefficient $\boldsymbol{\gamma}_0 = \boldsymbol{\gamma}_{10}$ or $\boldsymbol{\gamma}_{20}$ for $L_0 = 1$, and $\boldsymbol{\gamma}_0 = (\boldsymbol{\gamma}_{10}^{\mathrm{T}}, \boldsymbol{\gamma}_{20}^{\mathrm{T}})^{\mathrm{T}}$ for $L_0 = 2$.

For a candidate $\boldsymbol{\theta} = (\boldsymbol{\gamma}, \boldsymbol{\beta})$, we let $\{R_j^{(4)}(\boldsymbol{\gamma})\}_{j=1}^4$ be the four regimes under $\boldsymbol{\gamma}$, and denote $\mathcal{G} = \{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2\}$ and $\mathcal{B} = \{\boldsymbol{\beta}_1, \cdots, \boldsymbol{\beta}_4\}$. Then, the population of the LS criterion function based on the four-regime model is

$$\mathbb{M}(\boldsymbol{\theta}) = \mathbb{E}[\{Y - \sum_{j=1}^{4} \boldsymbol{X}^{\mathrm{T}} \boldsymbol{\beta}_j \mathbb{1}\{\boldsymbol{Z} \in R_j^{(4)}(\boldsymbol{\gamma})\}\}^2].$$

Suppose that when the data is generated from Model (E.1) with $K_0 \leq 4$, $\mathbb{M}(\boldsymbol{\theta})$ is minimized at $\boldsymbol{\theta}_* = (\boldsymbol{\gamma}_*^{\mathrm{T}}, \boldsymbol{\beta}_*^{\mathrm{T}})^{\mathrm{T}}$. Let $\mathcal{G}_* = \{\boldsymbol{\gamma}_{1*}, \boldsymbol{\gamma}_{2*}\}$ and $\mathcal{B}_* = \{\boldsymbol{\beta}_{1*}, \cdots, \boldsymbol{\beta}_{4*}\}$, representing the true parameters under the four-segment model. In the case of $K_0 = 4$, we have shown that $\boldsymbol{\theta}_* = \boldsymbol{\theta}_0$ in Proposition 1. Now we show that when $K_0 < 4$, the true parameters $\boldsymbol{\gamma}_0$ and $\boldsymbol{\beta}_0$ are elements of $\mathcal{G}_*$ and $\mathcal{B}_*$, respectively. That is, we are to show that $d(\boldsymbol{\gamma}_0, \mathcal{G}_*) = 0$ and $d(\boldsymbol{\beta}_{k0}, \mathcal{B}_*) = 0$ for $k = 1, 2$. Without loss of generality, we take $L_0 = 1$ and $K_0 = 2$ in this proof, which makes Model (E.1) to be the two-regime model (6.3) of the main paper. The proof for the other degenerated models can be shown similarly.

Note that

$$\begin{aligned}
\mathbb{M}(\boldsymbol{\theta}) &= \mathbb{E}[\{Y - \sum_{j=1}^{4} \boldsymbol{X}^{\mathrm{T}} \boldsymbol{\beta}_j \mathbb{1}\{\boldsymbol{Z} \in R_j^{(4)}(\boldsymbol{\gamma})\}\}^2] \\
&= \mathbb{E}[\varepsilon^2 + \{\sum_{k=1}^{2} \boldsymbol{X}^{\mathrm{T}} \boldsymbol{\beta}_{k0} \mathbb{1}\{\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0)\} - \sum_{j=1}^{4} \boldsymbol{X}^{\mathrm{T}} \boldsymbol{\beta}_j \mathbb{1}\{\boldsymbol{Z} \in R_j^{(4)}(\boldsymbol{\gamma})\}\}^2] \\
&= \mathbb{E}(\varepsilon^2) + \sum_{k=1}^{2} \sum_{j=1}^{4} \mathbb{E}[\{\boldsymbol{X}^{\mathrm{T}}(\boldsymbol{\beta}_{k0} - \boldsymbol{\beta}_j)\}^2 \mathbb{1}\{\boldsymbol{Z} \in R_k(\boldsymbol{\gamma}_0) \cap R_j^{(4)}(\boldsymbol{\gamma})\}] \\
&= \mathbb{E}(\varepsilon^2) + \sum_{k=1}^{2} \sum_{j=1}^{4} A_{k,j}(\boldsymbol{\theta}), \ \text{ say,}
\end{aligned} \tag{E.2}$$

where the second equality is due to $\mathbb{E}(\varepsilon | \boldsymbol{X}, \boldsymbol{Z}) = 0$. At $\boldsymbol{\theta} = \boldsymbol{\theta}_*$, it can be shown that $A_{k,j}(\boldsymbol{\theta}_*) = 0$ for any k, j. Hence $\mathbb{M}(\boldsymbol{\theta}_*) = \mathbb{E}(\varepsilon^2)$.

Suppose that $d(\boldsymbol{\gamma}_0, \mathcal{G}) \neq 0$, namely $\boldsymbol{\gamma}_1 \neq \boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_2 \neq \boldsymbol{\gamma}_0$. Then the true splitting hyperplane $H_0 : \boldsymbol{z}^{\mathrm{T}} \boldsymbol{\gamma}_0 = 0$ will partition through at least one region $R_j^{(4)}(\boldsymbol{\gamma})$ for $j \in \{1, \cdots, 4\}$. By Assumption S2 (i) we have $\mathbb{P}\left\{\boldsymbol{Z} \in R_1(\boldsymbol{\gamma}_0) \cap R_j^{(4)}(\boldsymbol{\gamma})\right\} > 0$ and $\mathbb{P}\left\{\boldsymbol{Z} \in R_2(\boldsymbol{\gamma}_0) \cap R_j^{(4)}(\boldsymbol{\gamma})\right\} > 0$. Therefore,

$$A_{1,j}(\boldsymbol{\theta}) \geq \lambda_0 \|\boldsymbol{\beta}_j - \boldsymbol{\beta}_{10}\|^2, \ \ A_{2,j}(\boldsymbol{\theta}) \geq \lambda_0 \|\boldsymbol{\beta}_j - \boldsymbol{\beta}_{20}\|^2$$

according to Assumption S2 (ii). Since $\boldsymbol{\beta}_{10} \neq \boldsymbol{\beta}_{20}$, either $A_{1,j}(\boldsymbol{\theta}) > 0$ or $A_{2,j}(\boldsymbol{\theta}) > 0$. Consequently, $\mathbb{M}(\boldsymbol{\theta}) \geq \mathbb{M}(\boldsymbol{\theta}_*) + A_{k,h}(\boldsymbol{\theta}) + A_{j,h}(\boldsymbol{\theta}) > \mathbb{M}(\boldsymbol{\theta}_*)$.

Suppose that $d(\boldsymbol{\gamma}_0, \mathcal{G}) = 0$, namely $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_0$ or $\boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_0$, while $d(\boldsymbol{\beta}_{k0}, \mathcal{B}) \neq 0$ for $k \in \{1, 2\}$. In such case, there exits $j \in \{1, \cdots, 4\}$ such that $R_j^{(4)}(\boldsymbol{\gamma}) \subset R_k(\boldsymbol{\gamma}_0)$. Hence

$$A_{k,j}(\boldsymbol{\theta}) = \mathbb{E}\left[\{\boldsymbol{X}_t^{\mathrm{T}}(\boldsymbol{\beta}_j - \boldsymbol{\beta}_{k0})\}^2 \mathbb{1}\{\boldsymbol{Z}_t \in R_k(\boldsymbol{\gamma}_0)\}\right] \geq \lambda_0 \|\boldsymbol{\beta}_j - \boldsymbol{\beta}_{k0}\|^2 > 0,$$

by Assumption S2 (ii). Therefore, $\mathbb{M}(\boldsymbol{\theta}) \geq \mathbb{M}(\boldsymbol{\theta}_*) + A_{k,j}(\boldsymbol{\theta}) > \mathbb{M}(\boldsymbol{\theta}_*)$.

Combining the two cases yields that

$$\mathbb{M}(\boldsymbol{\theta}) > \mathbb{M}(\boldsymbol{\theta}_*) \text{ for any } \boldsymbol{\theta} \in \Theta \tag{E.3}$$

if either $d(\boldsymbol{\gamma}_0, \mathcal{G}) \neq 0$ or $d(\boldsymbol{\beta}_{k0}, \mathcal{B}) \neq 0$ for some $k \in \{1, 2\}$. Therefore, $\boldsymbol{\theta}_*$ as the minimizer of $\mathbb{M}(\boldsymbol{\theta})$ must satisfy $d(\boldsymbol{\gamma}_0, \mathcal{G}_*) = 0$ and $d(\boldsymbol{\beta}_{k0}, \mathcal{B}_*) = 0$ for $k \in \{1, 2\}$.

Having established the minimizer of the least square criterion function under the population level, the rest of the proof for the convergence rate of the LS estimator under Assumptions S3 and S4, follows the similar arguments as in Appendix B for the four-regime case. □

E.2. Proof of Theorem 6.2.

PROOF. Suppose that the true model is given by (E.1) with the K_0 true regimes being $\{R_{k0}\}_{k=1}^{K_0}$ and the true regression coefficients are $\{\boldsymbol{\beta}_{k0}\}_{k=1}^{K_0}$ respectively. Let the estimated regimes and the estimated regression coefficients under the four-regime model be $\{\widehat{R}_j^{(4)}\}_{j=1}^4$ and $\{\widehat{\boldsymbol{\beta}}_j^{(4)}\}_{j=1}^4$, respectively.

For any $1 \leq K \leq 4$, let

$$\mathcal{C}_T(K) = \log\left(\frac{S_T(K)}{T}\right) + \frac{\lambda_T}{T} K,$$

where $\lambda_T \to \infty$ and $\lambda_T/T \to 0$ as $T \to \infty$, and

$$S_T(4) = \sum_{t=1}^{T}\left[Y_t - \sum_{k=1}^{4} \boldsymbol{X}_t^{\mathrm{T}} \widehat{\boldsymbol{\beta}}_k^{(4)} \mathbb{1}\left\{\boldsymbol{Z}_t \in \widehat{R}_k^{(4)}\right\}\right]^2.$$

For $1 \leq K \leq 3$, define recursively

$$S_T(K) = S_T(K+1) + D_T^{(K+1)}(\widehat{i}, \widehat{h}),$$

where $(\widehat{i}_{K+1}, \widehat{h}_{K+1}) = \arg\min_{\mathcal{A}_{k+1}} D_T^{(K+1)}(i, h)$ and

$$\begin{aligned} D_T^{(K)}(i,h) =& \min_{\boldsymbol{\beta} \in \mathbb{B}} \sum_{t=1}^{T} [Y_t - \boldsymbol{X}_t^{\mathrm{T}} \boldsymbol{\beta} \mathbb{1}\{\boldsymbol{Z}_t \in \widehat{R}_i^{(K)} \cup \widehat{R}_h^{(K)}\}]^2 \\ & - \sum_{t=1}^{T} [Y_t - \sum_{k=i,h} \boldsymbol{X}_t^{\mathrm{T}} \widehat{\boldsymbol{\beta}}_k^{(K)} \mathbb{1}\{\boldsymbol{Z}_t \in \widehat{R}_k^{(K)}\}]^2 \\ =:& S_{i,h}^{(K)} - T_{i,h}^{(K)}, \text{ say.} \end{aligned}$$

First, we claim that for $K \geq K_0$, for each $1 \leq h \leq K_0$, there exists an index set $\mathcal{Q}_h^{(K)} \subset \{1, \cdots, K\}$ such that

$$\mathbb{P}\left\{\boldsymbol{Z}_t \in R_{h0} \triangle \cup_{i \in \mathcal{Q}_h^{(K)}} \widehat{R}_i^{(K)}\right\} = O(T^{-1}) \text{ and } \max_{i \in \mathcal{Q}_h^{(K)}} \|\boldsymbol{\beta}_{h0} - \widehat{\boldsymbol{\beta}}_i^{(K)}\| = O_p(T^{-1/2}). \tag{E.4}$$

We will prove the claim recursively. Specifically, we are to show that if (E.4) holds for $K = \tilde{K}$, then it also holds for $K = \tilde{K} - 1$, by showing that the index pair for merged regimes (i, h) from the $\tilde{K}$-segments model to the $(\tilde{K} - 1)$-segments model satisfies $(i, h) \in \mathcal{Q}_k^{(\tilde{K})}$ for some $1 \le k \le \tilde{K}$.

We start with $K = 4$, where (E.4) is ensured by Theorem 6.1. For the case of $K = 3$, we now show that $D_T^{(4)}(i, h) < D_T^{(4)}(i', h')$ if $\{i, h\} \subset \mathcal{Q}_k^{(4)}$ for some $1 \le k \le K_0$ and $\{i', h'\} \not\subset \mathcal{Q}_k^{(4)}$ for any $1 \le k \le K_0$, which implies that the selected merged regimes leading to the submodel with $K = 3$ are $\widehat{R}_i^{(4)}$ and $\widehat{R}_h^{(4)}$ which are asymptotically contained in the same regime R_{k0}.

Case (1). If two indices $\{i_k, h_k\} \subset \mathcal{Q}_k^{(4)}$, with some standard algebra, we can obtain

$$D_T^{(4)}(i_k, h_k) = S_{i_k,h_k}^{(4)} - T_{i_k,h_k}^{(4)} = \mathbf{H}_T(\widehat{R}_{i_k}^{(4)})^{\mathrm{T}} \boldsymbol{\Xi}_T^{-1} \mathbf{H}_T(\widehat{R}_{i_k}^{(4)}),$$

where

$$\mathbf{H}_T(\widehat{R}_{i_k}^{(4)}) = \{\mathbf{I}_p - \mathbf{G}_T(\widehat{R}_i^{(4)}) \mathbf{G}_{k,T}^{-1}\} \sqrt{T} \mathbb{E}_T \left\{ \varepsilon_t \boldsymbol{X}_t \mathbb{1}(\boldsymbol{Z}_t \in \widehat{R}_{i_k}^{(4)}) \right\} \quad \text{and}$$

$$\boldsymbol{\Xi}_T = \mathbf{G}_T(\widehat{R}_{i_k}^{(4)}) - \mathbf{G}_T(\widehat{R}_{i_k}^{(4)}) \mathbf{G}_{k,T}^{-1} \mathbf{G}_T(\widehat{R}_{i_k}^{(4)}),$$

with $\mathbf{G}_T(\widehat{R}_{i_k}^{(4)}) = \mathbb{E}_T[\boldsymbol{X}_t \boldsymbol{X}_t^{\mathrm{T}} \mathbb{1}\{\boldsymbol{Z}_t \in \widehat{R}_{i_k}^{(4)}\}]$ and $\mathbf{G}_{k,T} = \mathbb{E}_T[\boldsymbol{X}_t \boldsymbol{X}_t^{\mathrm{T}} \mathbb{1}\{\boldsymbol{Z}_t \in R_{k0}\}]$ for each $1 \le k \le K_0$ and $i_k \in \mathcal{Q}_k^{(4)}$. Using the martingale central limit theorem and the uniform law of large numbers, it can be easily seen that

$$D_T^{(4)}(i_k, h_k) = O_p(1), \quad \text{if } \{i_k, h_k\} \subset \mathcal{Q}_k^{(4)} \text{ for each } 1 \le k \le K_0. \tag{E.5}$$

Case (2). If the two indices $\{i, h\} \not\subset \mathcal{Q}_k^{(4)}$ for any $1 \le k \le K_0$. Suppose that $i \in \mathcal{Q}_{\tilde{i},}^{(4)}$ and $h \in \mathcal{Q}_{\tilde{h}}^{(4)}$, for some $1 \le \tilde{i}, \tilde{h} \le K_0$. Then Theorem 6.1 implies that that $\mathbb{P}\{\boldsymbol{Z}_t \in \widehat{R}_i^{(4)} \setminus R_{\tilde{i}0}\} = O_p(1/T), \|\boldsymbol{\beta}_{\tilde{i}0} - \widehat{\boldsymbol{\beta}}_i\| = O_p(1/\sqrt{T})$, and the same consistency also holds for $\widehat{R}_h^{(4)}$ and $\widehat{\boldsymbol{\beta}}_h$. Then standard algebra leads to

$$T_{i,h}/T = \mathbb{E}_T[\varepsilon_t^2 \mathbb{1}\{\boldsymbol{Z}_t \in \widehat{R}_i^{(4)} \cup \widehat{R}_h^{(4)}\}] + o_p(1), \quad \text{and} \tag{E.6}$$

$$S_{i,h}/T = \mathbb{E}_T[\varepsilon_t^2 \mathbb{1}\{\boldsymbol{Z}_t \in \widehat{R}_i^{(4)} \cup \widehat{R}_h^{(4)}\}] + \boldsymbol{\delta}_{\tilde{i}\tilde{h},0}^{\mathrm{T}} \mathbf{G}_T(\widehat{R}_i^{(4)}) \mathbf{G}_T(\widehat{R}_{i\cup h}^{(4)})^{-1} \mathbf{G}_T(\widehat{R}_h^{(4)}) \boldsymbol{\delta}_{\tilde{i}\tilde{h},0} + o_p(1),$$

where $\mathbf{G}_T(\widehat{R}_{i\cup h}^{(4)}) = \mathbb{E}_T[\boldsymbol{X}_t \boldsymbol{X}_t^{\mathrm{T}} \mathbb{1}\{\boldsymbol{Z}_t \in \widehat{R}_i^{(4)} \cup \widehat{R}_h^{(4)}\}]$. By Assumption S2 and the ULLN, the smallest eigenvalue of $\mathbf{G}_T(\widehat{R}_i^{(4)}) \mathbf{G}_T(\widehat{R}_{i\cup h}^{(4)})^{-1} \mathbf{G}_T(\widehat{R}_h^{(4)})$ is asymptotically bounded away from some constant $\lambda_1 > 0$. Since $\boldsymbol{\delta}_{\tilde{i}\tilde{h},0} = \boldsymbol{\beta}_{\tilde{i}0} - \boldsymbol{\beta}_{\tilde{h}0} \neq \mathbf{0}$ as required in Assumption S2, from (E.6) we obtain

$$D_T^{(4)}(i, h) = S_{i,h} - T_{i,h} = O_p(T), \quad \text{if } \{i, h\} \not\subset \mathcal{Q}_k^{(4)} \text{ for any } 1 \le k \le K_0. \tag{E.7}$$

This together with (E.5) and (E.7) implies that the optimal regime merger from $K = 4$ to $K = 3$ is the pair of regimes that are contained in the same $\mathcal{Q}_k^{(4)}$ for some $1 \le k \le K_0$. Hence, (E.4) with $K = 3$ is verified. Using the same argument the claim (E.4) with $K = 2$ and 1 can also be established, respectively, provided that $K \ge K_0$.

(E.4) implies that with some relabelling,

$$\mathbb{P}\{\widehat{R}_k^{(K_0)} \triangle R_{k0}\} = O(T^{-1}) \text{ and } \|\boldsymbol{\beta}_{k0} - \widehat{\boldsymbol{\beta}}_k\| = O_p(T^{-1/2}), \tag{E.8}$$

for each $1 \le k \le K_0$. which reveals that the back-elimination procedure consistently estimates the true model, if it can be shown that $\mathbb{P}(\widehat{K} = K_0) \to 1$ as $T \to \infty$.

We now show that $\mathbb{P}\{\mathcal{C}_T(K) < \mathcal{C}_T(K_0)\} \to 0$ when $K \neq K_0$, which ensures the model selection consistency.

(i) First, if $K < K_0$, by the definition of $\mathcal{C}(K)$, we have

$$\mathbb{P}\{\mathcal{C}_T(K) < \mathcal{C}_T(K_0)\} = \mathbb{P}\left\{\log\left(\frac{S_T(K)}{S_T(K_0)}\right) < \frac{\lambda_T(K_0 - K)}{T}\right\}. \tag{E.9}$$

As $\lambda_T/T \to 0$, to show the above probability converges to 0, it suffices to show that $\mathbb{P}\{S_T(K) > S_T(K_0)\} \to 1$. Note that (E.4) means that $|\mathcal{Q}_h^{(K_0)}| = 1$ for each $1 \le h \le K_0$. Similar to (E.7), it is straightforward to show that $D_T^{(K)}(\hat{i}_K, \hat{h}_K) > 0$ for each $1 \le K \le K_0$, meaning that any under-segmented models have increased sum of squared residuals. As $S_T(K) - S_T(K_0) = \sum_{k=K+1}^{K_0} D_T^{(k)}(\hat{i}_k, \hat{h}_k)$, we have $\mathbb{P}\{S_T(K) > S_T(K_0)\} \to 1$, which implies (E.9) converges to 0 as $\lambda_T/T \to 0$.

(ii) If $K > K_0$, meaning that the K-regime model is over-segmented, we have

$$\begin{aligned}\mathbb{P}\{\mathcal{C}_T(K) < \mathcal{C}_T(K_0)\} &= \mathbb{P}\left\{\log\left(\frac{S_T(K_0)}{S_T(K)}\right) > \frac{\lambda_T(K - K_0)}{T}\right\} \\ &= \mathbb{P}\left\{\frac{S_T(K_0) - S_T(K)}{S_T(K)/T} > T\left(e^{\frac{\lambda_T(K-K_0)}{T}} - 1\right)\right\},\end{aligned} \tag{E.10}$$

and $S_T(K_0) = S_T(K) + \sum_{k=K_0+1}^{K} D_T^{(k)}(\hat{i}_k, \hat{h}_k)$. Because of (E.5) we have $S_T(K_0) - S_T(K) = O_p(1)$. In addition, $S_T(K_0)/T = \mathbb{E}_T(\varepsilon_t^2) = O_p(1)$. By the Taylor expansion, $T\left(e^{\frac{\lambda_T(K-K_0)}{T}} - 1\right) = O(\lambda_T) \to \infty$. Hence, the probability in (E.10) converges to 0, .

Combining Cases (i) can (ii), we have $\mathbb{P}\{\mathcal{C}_T(K) < \mathcal{C}_T(K_0)\} \to 0$ if $K \neq K_0$. Since $\widehat{K} = \arg\min_{1 \le K \le 4} \mathcal{C}(K)$ and $1 \le K_0 \le 4$, it implies that $\mathbb{P}(\widehat{K} = K_0) \to 1$ as $T \to \infty$. This together with (E.8) completes the proof. □

APPENDIX F: AUXILIARY ASSUMPTIONS

F.1. Sufficient conditions for some assumptions. In this part, we provide some sufficient conditions for Assumptions 2.(i), 3.(ii), and 4.(i).

ASSUMPTION S1. (i) For each $l \in \{1, 2\}$, let $q_l = \boldsymbol{Z}^{\mathrm{T}}\boldsymbol{\gamma}_{l0}$. There exists some $j \in \{1, \cdots, d_l\}$, such that for almost surely $\boldsymbol{Z}_{-1,l}$, the conditional density $f_{q_l|\boldsymbol{Z}_{-1,l}}(q)$ is continuous at $q = 0$ and $f_{q_{l,t}|\boldsymbol{Z}_{-1,l}}(0) \ge c_0$ for almost surely $\boldsymbol{Z}_{-1,l}$, where c_0 is a positive constant.

(ii) For each $l \in \{1, 2\}$, there exists $c_1 > 0$ and $j \in [d_l]$ such that the conditional density $f_{q_l|\boldsymbol{Z}_{-1,l}}(q|\boldsymbol{z}) < c_1$ for almost surely $q \in \mathbb{R}$ and $\boldsymbol{z} \in \mathcal{Z}_{-1,l}$, where $\mathcal{Z}_{-1,l}$ is the support for the distribution of $\boldsymbol{Z}_{-1,l}$ and is a compact set in $\mathbb{R}^{d_l - 1}$.

(iii) For each $l \in \{1, 2\}$, there exist some $j_l \in [d_l]$ and $c_2 > 0$ such that the conditional density $f_{(q_1,q_2)|(\boldsymbol{Z}_{-j_1,1},\boldsymbol{Z}_{-j_2,2})}(q_1, q_2|\boldsymbol{z}_1, \boldsymbol{z}_2) < c_2$ for almost surely $(q_1, q_2) \in \mathbb{R}^2$ and $(\boldsymbol{z}_1, \boldsymbol{z}_2) \in \mathcal{Z}_{-j_1,1} \times \mathcal{Z}_{-j_2,2}$, where $\mathcal{Z}_{-j_l,l}$ is the support for the distribution of $\boldsymbol{Z}_{-j_l.l}$ and is a compact set in $\mathbb{R}^{d_l - 1}$ for each $l \in \{1, 2\}$.

The following lemma shows that Assumption S1 implies Assumptions 2.(i), 3.(ii), 4.(i) and 4.(iii).

LEMMA F.1. *(i) Under Assumption S1 (i), there exists some constant $\delta_1 > 0$, if $\epsilon < \delta$, then $\mathbb{P}(|q_l| < \epsilon|\boldsymbol{Z}_{-1,l}) \ge c_0\epsilon/2$ almost surely, implying Assumptions 2.(i) and 4.(i).*

(ii) Under Assumption S1 (ii), there exist some positive constants δ_2 and c_1 such that if $\gamma_1, \gamma_2 \in \mathcal{N}(\gamma_{l0}; \delta_2)$, then $|\mathbb{P}(\boldsymbol{Z}_l^{\mathrm{T}}\gamma_1 < 0) - \mathbb{P}(\boldsymbol{Z}_l^{\mathrm{T}}\gamma_2 < 0)| \le c_3\|\gamma_1 - \gamma_2\|$, which ensures Assumptions 3.(ii).

(iii) Under Assumption S1 (iii),there exist some positive constants δ_3 and c_2 such that if $\gamma_1, \gamma_2 \in \mathcal{N}(\gamma_{10}; \delta_0)$ and $\gamma_3, \gamma_4 \in \mathcal{N}(\gamma_{20}, \delta_2)$, then $\mathbb{P}(\boldsymbol{Z}_1^{\mathrm{T}}\gamma_1 < 0 < \boldsymbol{Z}_1^{\mathrm{T}}\gamma_2, \boldsymbol{Z}_2^{\mathrm{T}}\gamma_3 < 0 < \boldsymbol{Z}_2^{\mathrm{T}}\gamma_4) \le c_2\|\gamma_1 - \gamma_2\|\|\gamma_3 - \gamma_4\|$, which ensures Assumptions 4.(iii).

PROOF. (i) The continuity of $f_{q_{l,t}|\boldsymbol{Z}_{-j,l}}(q)$ at $q = 0$ in Assumption S1 (i) implies that there exists $\delta_1 > 0$ such that $f_{q_l|\boldsymbol{Z}_{-1,l}}(|q|) \ge f_{q_l|\boldsymbol{Z}_{-1,l}}(0) - c_1/2 \ge c_1/2$. The assertion then follows from $\mathbb{P}(|q_l| < \epsilon|\boldsymbol{Z}_{-i,l}) = \int_{-\epsilon}^{\epsilon} f_{q_l|\boldsymbol{Z}_{-1,l}}(q)dq$.

(ii) Let $\Delta_l(\gamma) = \boldsymbol{Z}_l^{\mathrm{T}}(\gamma_{l0} - \gamma)$. Then for any $\gamma_1, \gamma_2 \in \mathcal{N}(\gamma_{l0}; \delta_1)$, where $\delta_1 < \delta_0/B$,

$$\mathbb{P}\{\boldsymbol{Z}_l^{\mathrm{T}}\gamma_1 > 0 > \boldsymbol{Z}_l^{\mathrm{T}}\gamma_2\} = \mathbb{P}\{\Delta_l(\gamma_1) < q_l < \Delta_l(\gamma_2)\} = E_{\boldsymbol{Z}_{-1,l}}\left\{\int_{\Delta_l(\gamma_1)}^{\Delta_l(\gamma_2)} f_{q_l|\boldsymbol{Z}_{-1,l}}(q)dq\right\}.$$

Let $M > 0$ be the constant such that $\|\boldsymbol{z}\|_\infty < M$ for all $\boldsymbol{z} \in \mathcal{Z}_{-j,l}$ and let $\delta_1 = \delta_0/M$, which ensures $\|\Delta_l(\gamma)\|_\infty \le \delta_0$ whenever $\gamma \in \mathcal{N}(\gamma_{l0}; \delta_1)$. It is then straightforward to see that $\mathbb{P}\{\boldsymbol{Z}_l^{\mathrm{T}}\gamma_1 > 0 > \boldsymbol{Z}_l^{\mathrm{T}}\gamma_2\} \le c_1 M\|\gamma_1 - \gamma_2\|$. Similary, $\mathbb{P}\{\boldsymbol{Z}_l^{\mathrm{T}}\gamma_1 < 0 < \boldsymbol{Z}_l^{\mathrm{T}}\gamma_2\}$ can be bounded in the same way. Since $|\mathbb{P}(\boldsymbol{Z}_l^{\mathrm{T}}\gamma_1 < 0) - \mathbb{P}(\boldsymbol{Z}_l^{\mathrm{T}}\gamma_2 < 0)| = \mathbb{P}\{\boldsymbol{Z}_l^{\mathrm{T}}\gamma_1 > 0 > \boldsymbol{Z}_l^{\mathrm{T}}\gamma_2\} + \mathbb{P}\{\boldsymbol{Z}_l^{\mathrm{T}}\gamma_1 < 0 < \boldsymbol{Z}_l^{\mathrm{T}}\gamma_2\}$, the desired result follows.

(iii) It follows from the similar argument as in (ii) and thus is omitted. □

F.2. Assumptions for degenerated models. The following assumption adapts Assumptions 2-4 of the main article for the segmented regression with the number of regimes $K_0 = 4$ and the number of splitting hyperplanes $L_0 = 2$ to the degenerated models with $1 \le K_0 \le 3$ and $0 \le L_0 \le 2$, which include Model (6.1)–(6.5) in the main article. Let $(Y, \boldsymbol{X}, \boldsymbol{Z}) \sim \mathbb{P}_0$. Suppose the data generated from a model

$$Y = \sum_{k=1}^{K_0} \boldsymbol{X}^{\mathrm{T}}\boldsymbol{\beta}_{k0}\mathbb{1}\{\boldsymbol{Z} \in R_k(\gamma_0)\} + \varepsilon, \tag{F.1}$$

where the number of regimes $1 \le K_0 \le 3$ and the number of splitting hyperplanes $0 \le L_0 \le 2$. In particular, $R_k(\gamma_0) = \mathcal{Z}_1 \times \mathcal{Z}_2$ for the global linear model ($K_0 = 1$), the splitting coefficient $\gamma_0 = \gamma_{10}$ or γ_{20} when $L_0 = 1$, and $\gamma_0 = (\gamma_{10}^{\mathrm{T}}, \gamma_{20}^{\mathrm{T}})^{\mathrm{T}}$ when $L_0 = 2$. We use $\mathcal{L}_0 \subset \{1, 2\}$ to indicate the indices of the splitting hyperplanes. For instance, if the true model has two hyperplanes then $\mathcal{L}_0 = \{1, 2\}$; and if it has only one hyperplane $H_{20} = \{\boldsymbol{z}_2^{\mathrm{T}}\gamma_{20} = 0\}$ then $\mathcal{L}_0 = \{2\}$. The following assumptions are needed for Theorem 6.1.

ASSUMPTION S2. For each $i \in \mathcal{L}_0$ and $k, h \in \{1, \cdots, K_0\}$, the following conditions hold. (i) If $L_0 = 2$, then $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ are not identically distributed. (ii) There exists a $j \in [d_i]$ such that $\mathbb{P}(|q_i| \le \epsilon|\boldsymbol{Z}_{-j,i}) > 0$ for almost surely $\boldsymbol{Z}_{-j,i}$ for any $\epsilon > 0$, where $\boldsymbol{Z}_{-j,i}$ is the vector after excluding $\boldsymbol{Z}_i$'s jth element. Without loss of generality, assume $j = 1$. (iii) For any $\gamma \in \Gamma_1 \times \Gamma_2$, if $\mathbb{P}\{\boldsymbol{Z} \in R_k(\gamma_0) \cap R_h(\gamma)\} > 0$, then the smallest eigenvalue of $\mathbb{E}\{\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}|\boldsymbol{Z} \in R_k(\gamma_0) \cap R_h(\gamma)\} \ge \lambda_0$ for some constant $\lambda_0 > 0$. (iv) If $(k, h) \in \mathcal{S}(i)$, then $\|\boldsymbol{\beta}_{k0} - \boldsymbol{\beta}_{h0}\| > c_0$ for some constant $c_0 > 0$, where $\mathcal{S}(i)$ is defined in (3).

ASSUMPTION S3. (i) $\mathbb{E}(Y^4) < \infty$, $\mathbb{E}(\|\boldsymbol{X}\|^4) < \infty$ and $\max_{i \in \mathcal{L}_0} \mathbb{E}(\|\boldsymbol{Z}_i\|) < \infty$. (ii) For each $i \in \mathcal{L}_0$, $\mathbb{P}(\boldsymbol{Z}_i^{\mathrm{T}}\gamma_1 < 0 < \boldsymbol{Z}_i^{\mathrm{T}}\gamma_2) \le c_1\|\gamma_1 - \gamma_2\|$ if $\gamma_1, \gamma_2, \in \mathcal{N}(\gamma_{i0}; \delta_0)$, for some constants $\delta_0, c_1 > 0$.

ASSUMPTION S4. (i) For $i \in \mathcal{L}_0$, there exist constants $\delta_1, c_2 > 0$ such that if $\epsilon \in (0, \delta_1)$ then $\mathbb{P}(|q_i| < \epsilon | \boldsymbol{Z}_{-1,i}) \geq c_2\epsilon$ almost surely. (ii) For $i \in \mathcal{L}_0$, there exists a neighborhood $\mathcal{N}_i = \mathcal{N}(\boldsymbol{\gamma}_{i0}; \delta_2)$ for some $\delta_2 > 0$, such that $\inf_{\boldsymbol{\gamma} \in \mathcal{N}_i} \mathbb{E}(\|\boldsymbol{X}\| | \boldsymbol{Z}_i^{\mathrm{T}} \boldsymbol{\gamma} = 0) > 0$ almost surely. (iii) If $L_0 = 2$, then $\mathbb{P}(\boldsymbol{Z}_1^{\mathrm{T}} \boldsymbol{\gamma}_1 < 0 < \boldsymbol{Z}_1^{\mathrm{T}} \boldsymbol{\gamma}_2, \boldsymbol{Z}_2^{\mathrm{T}} \boldsymbol{\gamma}_3 < 0 < \boldsymbol{Z}_2^{\mathrm{T}} \boldsymbol{\gamma}_4) \leq c_3 \|\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2\| \|\boldsymbol{\gamma}_3 - \boldsymbol{\gamma}_4\|$ for some constant $c_3 > 0$ if $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \mathcal{N}_1$ and $\boldsymbol{\gamma}_3, \boldsymbol{\gamma}_4 \in \mathcal{N}_2$. (iv) For some constant $r > 8$, $\sup_{\boldsymbol{\gamma} \in \mathcal{N}_i} \mathbb{E}(\|\boldsymbol{X}\|^r | \boldsymbol{Z}_i^{\mathrm{T}} \boldsymbol{\gamma} = 0) < \infty$ and $\sup_{\boldsymbol{\gamma} \in \mathcal{N}_i} \mathbb{E}(\varepsilon^r | \boldsymbol{Z}_i^{\mathrm{T}} \boldsymbol{\gamma} = 0) < \infty$ almost surely.

APPENDIX G: EXTENSION TO GENERAL SEGMENTED REGRESSIONS

In this section, we discuss the extension of the proposed four-regime segmented regression to general segmented regressions with more than $L = 2$ splits. The range of numbers of regimes split by L hyperplanes is presented by the following result, whose proof can be seen in Buck (1943).

THEOREM G.1. *Suppose that there are $L \geq 1$ hyperplanes $H_l = \{\boldsymbol{z} \in \mathcal{Z} : \boldsymbol{z}^{\mathrm{T}} \boldsymbol{\gamma}_i = 0\}_{i=1}^{L}$. Then the number of regimes K split by these L hyperplanes satisfies*

$$L - 1 \leq K \leq \sum_{i=0}^{\min(L,d)} \binom{L}{i}. \tag{G.1}$$

REMARK G.1. The above bound in sharp and can be attained in general hyperplane arrangement (see e.g., Orlik and Terao, 2013). It reveals the challenges in the general segmented linear regressions. First, in the large or high dimensional setting where $d > L$, the right-hand of (G.1) becomes 2^L. It implies that each possible combination of the signs of the $\{\boldsymbol{z}^{\mathrm{T}} \boldsymbol{\gamma}_i, 1 \leq i \leq L\}$ determines a specific region. Under such a circumstance, the computation burdens will be quite high in both optimization and model selection to select among the models with $1 \leq K \leq 2^L$. Moreover, the increase of K can bring more risk of overfitting.

On the other hand, under the regime where $d < L$, the maximum number of region $K_{\max}$ is $\sum_{i=0}^{d} \binom{L}{i} = O(L^d)$. The main difficulty is in specifying the model form of segmented models , since it can be challenging to know which hyperplanes constitute the boundaries of each regime due to the complications of hyperplane arrangements. One possible solution is to via some data-driven method to determine the boundaries of each regime, while it brings more computational complexity and requires further studies.

APPENDIX H: ADDITIONAL SIMULATION RESULTS

H.1. Simulations under models with less than four regimes. This section presents results for the estimation based on the four-regime model when the underlying models were degenerated with less than four regimes. The true parameters for the degenerated were specified in Section 7.2 of the main paper. The data generating processes for $\{\boldsymbol{X}_t, \boldsymbol{Z}_{1,t}, \boldsymbol{Z}_{2,t}, \varepsilon_t\}_{t=1}^{T}$ included three the independence, the AR(0.2) and the MA(0.2) settings as that in Section 7.1 of the main paper. Table S2 summarizes the empirical averages of the L_2-distance between the sets of the true parameters and their estimates under the four-regime model: $D(\mathcal{G}_0, \widehat{\mathcal{G}})$ and $D(\mathcal{B}_0, \widehat{\mathcal{B}})$. In addition, to evaluate the cost of not knowing the number of the underlying regimes, we also estimated $\boldsymbol{\gamma}_0$ and $\boldsymbol{\beta}_0$ in the so-called oracle setting, in which the three degenerated models were known to have three or two regimes and the parameters were estimated by the LS estimators of the corresponding models, denoted by $\widehat{\boldsymbol{\gamma}}^{\mathrm{3REG}}, \widehat{\boldsymbol{\beta}}^{\mathrm{3REG}}$ and $\widehat{\boldsymbol{\gamma}}^{\mathrm{2REG}}, \widehat{\boldsymbol{\beta}}^{\mathrm{2REG}}$, respectively. The three-regime LS estimators were obtained via a new MIQP algorithm presented in Appendix C of the SM, while $\widehat{\boldsymbol{\beta}}^{\mathrm{2REG}}$ of the two-regime estimators were calculated by the algorithm of Lee et al. (2021).

Table S2 shows that the estimation errors as reflected by the distance measures $D(\mathcal{G}_0, \widehat{\mathcal{G}})$ and $D(\mathcal{B}_0, \widehat{\mathcal{B}})$ reduced as the sample sizes T was increased, confirming that the parameters of the degenerated models could be consistently estimated by the four-regime model. By comparing $D(\mathcal{G}_0, \widehat{\mathcal{G}})$ with $\|\boldsymbol{\gamma}_0 - \widehat{\boldsymbol{\gamma}}^{3\text{REG}}\|$ and $\|\boldsymbol{\gamma}_0 - \widehat{\boldsymbol{\gamma}}^{2\text{REG}}\|$ in Table S2, we found that the estimation errors for $\boldsymbol{\gamma}_0$ based on the four-regime model were about the same as those of $\widehat{\boldsymbol{\gamma}}^{3\text{REG}}$ or $\widehat{\boldsymbol{\gamma}}^{2\text{REG}}$, respectively, meaning that the four-regime estimators achieved similar level of accuracy as the estimators from the models with correctly specified number of regimes, for the boundary coefficient estimation. The reason is that the four-regime estimator can efficiently use the data points located near the underlying boundaries as $\widehat{\boldsymbol{\gamma}}^{3\text{REG}}$ or $\widehat{\boldsymbol{\gamma}}^{2\text{REG}}$ did. On the other hand, Table S2 shows that the estimation accuracy for the regression coefficients based on the four-regime model were inferior to the estimators based on the models with the true number of regimes when the sample size was small. This was expected since the four-regime based estimation made redundant regime partitions, and hence did not effectively used the subsample belonged to the same underlying regime.

TABLE S2

Empirical average $D(\mathcal{G}_0, \widehat{\mathcal{G}}), D(\mathcal{B}_0, \widehat{\mathcal{B}})$, which represent the L_2 distance between the set of true parameters and their estimates by the four-regime model, and $\|\boldsymbol{\beta}_0 - \widehat{\boldsymbol{\beta}}^{3REG}\|_2, \|\boldsymbol{\gamma}_0 - \widehat{\boldsymbol{\gamma}}^{3REG}\|_2$, or $\|\boldsymbol{\beta}_0 - \widehat{\boldsymbol{\beta}}^{2REG}\|_2$ and $\|\boldsymbol{\gamma}_0 - \widehat{\boldsymbol{\gamma}}^{2REG}\|_2$ (multiplied by 10) under the independent (IND), auto-regressive (AR) and moving average (MA) settings for $\{\boldsymbol{X}_t^0, \boldsymbol{Z}_{1,t}^0, \boldsymbol{Z}_{2,t}^0\}_{t=1}^T$ of the three-regime model (a.2) and the two-regime model (b). The numbers inside the parentheses are the standard errors of the simulated averages.

T	IND				AR				MA			
Three-regime model (a.1)												
	$D(\mathcal{G}_0, \widehat{\mathcal{G}})$	$D(\mathcal{B}_0, \widehat{\mathcal{B}})$	$\widehat{\boldsymbol{\gamma}}^{3REG}$	$\widehat{\boldsymbol{\beta}}^{3REG}$	$D(\mathcal{G}_0, \widehat{\mathcal{G}})$	$D(\mathcal{B}_0, \widehat{\mathcal{B}})$	$\widehat{\boldsymbol{\gamma}}^{3REG}$	$\widehat{\boldsymbol{\beta}}^{3REG}$	$D(\mathcal{G}_0, \widehat{\mathcal{G}})$	$D(\mathcal{B}_0, \widehat{\mathcal{B}})$	$\widehat{\boldsymbol{\gamma}}^{3REG}$	$\widehat{\boldsymbol{\beta}}^{3REG}$
200	0.55	4.53	0.53	4.33	0.67	4.14	0.61	3.96	0.68	4.29	0.66	3.99
	0.26	1.24	0.22	0.95	0.35	0.82	0.33	0.84	0.30	1.15	0.27	0.84
400	0.30	3.09	0.32	2.95	0.30	2.85	0.32	2.73	0.30	2.84	0.31	2.74
	0.18	0.72	0.17	0.61	0.15	0.67	0.17	0.57	0.17	0.64	0.16	0.60
800	0.14	2.24	0.16	2.15	0.15	1.92	0.15	2.01	0.15	1.96	0.15	1.95
	0.07	0.51	0.06	0.48	0.08	0.47	0.08	0.45	0.06	0.37	0.05	0.37
1600	0.08	1.49	0.08	1.48	0.08	1.32	0.07	1.31	0.07	1.38	0.07	1.38
	0.04	0.35	0.04	0.34	0.04	0.28	0.04	0.27	0.04	0.28	0.04	0.27
Three-regime model (a.2)												
T	IND				AR				MA			
	$D(\mathcal{G}_0, \widehat{\mathcal{G}})$	$D(\mathcal{B}_0, \widehat{\mathcal{B}})$	$\widehat{\boldsymbol{\gamma}}^{3REG}$	$\widehat{\boldsymbol{\beta}}^{3REG}$	$D(\mathcal{G}_0, \widehat{\mathcal{G}})$	$D(\mathcal{B}_0, \widehat{\mathcal{B}})$	$\widehat{\boldsymbol{\gamma}}^{3REG}$	$\widehat{\boldsymbol{\beta}}^{3REG}$	$D(\mathcal{G}_0, \widehat{\mathcal{G}})$	$D(\mathcal{B}_0, \widehat{\mathcal{B}})$	$\widehat{\boldsymbol{\gamma}}^{3REG}$	$\widehat{\boldsymbol{\beta}}^{3REG}$
200	2.08	4.91	2.09	4.64	2.29	4.58	2.27	4.23	2.21	4.53	2.25	4.28
	(1.52)	(1.09)	(1.58)	(1.01)	(1.64)	(1.08)	(1.70)	(0.95)	(1.61)	(1.05)	(1.59)	(0.97)
400	1.00	3.35	1.03	3.20	1.12	3.04	1.13	2.91	1.12	3.06	1.10	2.91
	(0.85)	(0.73)	(0.87)	(0.71)	(0.81)	(0.63)	(0.80)	(0.63)	(0.82)	(0.67)	(0.78)	(0.63)
800	0.49	2.31	0.48	2.26	0.53	2.08	0.51	1.98	0.49	2.14	0.49	2.06
	(0.35)	(0.47)	(0.35)	(0.46)	(0.39)	(0.45)	(0.38)	(0.44)	(0.35)	(0.44)	(0.36)	(0.43)
1600	0.26	1.62	0.26	1.58	0.24	1.48	0.24	1.44	0.24	1.51	0.23	1.47
	(0.18)	(0.33)	(0.18)	(0.33)	(0.16)	(0.31)	(0.17)	(0.29)	(0.17)	(0.31)	(0.17)	(0.30)
Two-regime model												
T	IND				AR				MA			
	$D(\mathcal{G}_0, \widehat{\mathcal{G}})$	$D(\mathcal{B}_0, \widehat{\mathcal{B}})$	$\widehat{\boldsymbol{\gamma}}^{2REG}$	$\widehat{\boldsymbol{\beta}}^{2REG}$	$D(\mathcal{G}_0, \widehat{\mathcal{G}})$	$D(\mathcal{B}_0, \widehat{\mathcal{B}})$	$\widehat{\boldsymbol{\gamma}}^{2REG}$	$\widehat{\boldsymbol{\beta}}^{2REG}$	$D(\mathcal{G}_0, \widehat{\mathcal{G}})$	$D(\mathcal{B}_0, \widehat{\mathcal{B}})$	$\widehat{\boldsymbol{\gamma}}^{2REG}$	$\widehat{\boldsymbol{\beta}}^{2REG}$
200	0.55	3.25	0.54	2.85	0.59	2.95	0.60	2.54	0.64	3.26	0.64	2.59
	(0.44)	(0.83)	(0.43)	(0.74)	(0.45)	(0.78)	(0.48)	(0.68)	(0.49)	(0.80)	(0.48)	(0.68)
400	0.30	2.28	0.30	1.97	0.29	2.10	0.31	1.78	0.28	2.31	0.31	1.83
	(0.24)	(0.59)	(0.23)	(0.49)	(0.23)	(0.49)	(0.22)	(0.46)	(0.20)	(0.54)	(0.21)	(0.49)
800	0.14	1.69	0.14	1.41	0.14	1.25	0.15	1.23	0.15	1.49	0.14	1.29
	(0.10)	(0.43)	(0.11)	(0.35)	(0.12)	(0.32)	(0.13)	(0.32)	(0.11)	(0.36)	(0.11)	(0.32)
1600	0.07	1.02	0.07	0.97	0.06	0.93	0.07	0.88	0.07	0.94	0.07	0.90
	(0.05)	(0.27)	(0.06)	(0.25)	(0.05)	(0.23)	(0.05)	(0.22)	(0.05)	(0.24)	(0.05)	(0.23)
Global linear model												
T	IND				AR				MA			
		$D(\mathcal{B}_0, \widehat{\mathcal{B}})$		$\widehat{\boldsymbol{\beta}}^{GLR}$		$D(\mathcal{B}_0, \widehat{\mathcal{B}})$		$\widehat{\boldsymbol{\beta}}^{GLR}$		$D(\mathcal{B}_0, \widehat{\mathcal{B}})$		$\widehat{\boldsymbol{\beta}}^{GLR}$
200		1.81		1.33		1.48		1.22		1.87		1.19
		(0.67)		(0.49)		(0.55)		(0.47)		(0.62)		(0.45)
400		1.23		0.92		1.02		0.87		1.24		0.86
		(0.44)		(0.33)		(0.37)		(0.33)		(0.46)		(0.33)
800		0.83		0.69		0.78		0.59		0.85		0.64
		(0.23)		(0.23)		(0.27)		(0.22)		(0.30)		(0.22)
1600		0.62		0.46		0.51		0.43		0.54		0.43
		(0.24)		(0.18)		(0.18)		(0.15)		(0.18)		(0.16)

To gain further insights on the performances of the four-regime estimates under the degenerated models, we investigated the simulation results in more details by comparing adjacent estimated regression coefficients. Figure S1 displays the box plots of the squared distances between the estimated adjacent regression coefficients $\|\widehat{\boldsymbol{\beta}}_j - \widehat{\boldsymbol{\beta}}_k\|^2$ where the underlying samples were generated from the three-regime model (a.2). Figure S1 shows that as the sample size T was increased, $\|\widehat{\boldsymbol{\beta}}_1 - \widehat{\boldsymbol{\beta}}_2\|^2$, $\|\widehat{\boldsymbol{\beta}}_2 - \widehat{\boldsymbol{\beta}}_3\|^2$ and $\|\widehat{\boldsymbol{\beta}}_4 - \widehat{\boldsymbol{\beta}}_1\|^2$ converged to $\|\boldsymbol{\beta}_{10} - \boldsymbol{\beta}_{20}\|^2$, $\|\boldsymbol{\beta}_{20} - \boldsymbol{\beta}_{30}\|^2$ and $\|\boldsymbol{\beta}_{30} - \boldsymbol{\beta}_{10}\|^2$, respectively, while $\|\widehat{\boldsymbol{\beta}}_3 - \widehat{\boldsymbol{\beta}}_4\|^2$ decreased to 0, indicating that $\widehat{\boldsymbol{\beta}}_1$ and $\widehat{\boldsymbol{\beta}}_2$ were consistent estimates of $\boldsymbol{\beta}_{10}$ and $\boldsymbol{\beta}_{20}$, respectively, and both $\widehat{\boldsymbol{\beta}}_3$ and $\widehat{\boldsymbol{\beta}}_4$ converged to $\boldsymbol{\beta}_{30}$. Similar results for the two-regime model are also shown in Figure S2, which reveals that the estimated regression coefficients under the four-regime model could still provide consistent estimates to the underlying coefficients of the degenerated models.

Fig S1: Box plots for the squared distances of the estimated adjacent regression coefficient for the three-regime model (a.2). The red dashed lines indicate the squared distances of the true regression coefficients for the adjacent estimated regimes.

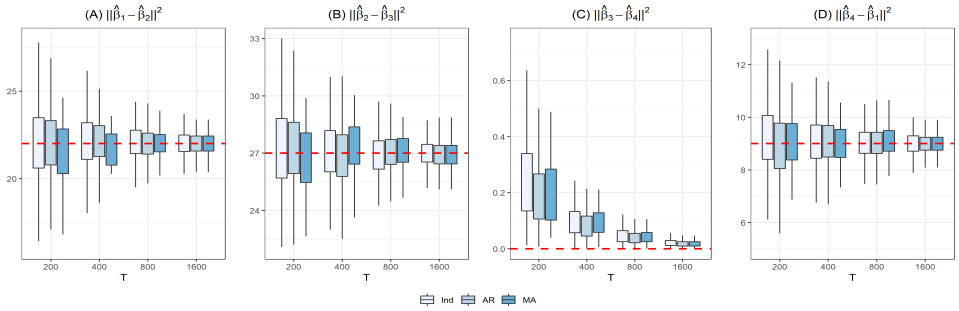


Fig S2: Box plots for the squared distances of the estimated adjacent regression coefficient for the two-regime model. The red dashed lines indicate the squared distances of the true regression coefficients of the three-regression model, with $\|\boldsymbol{\beta}_{10} - \boldsymbol{\beta}_{20}\|^2 = 22$.

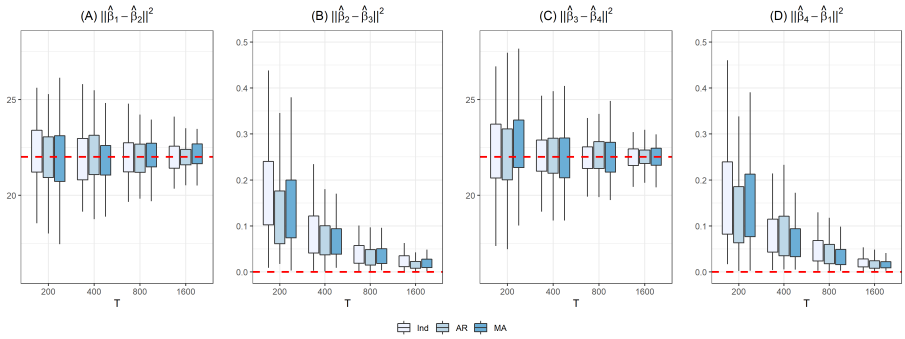


TABLE S3

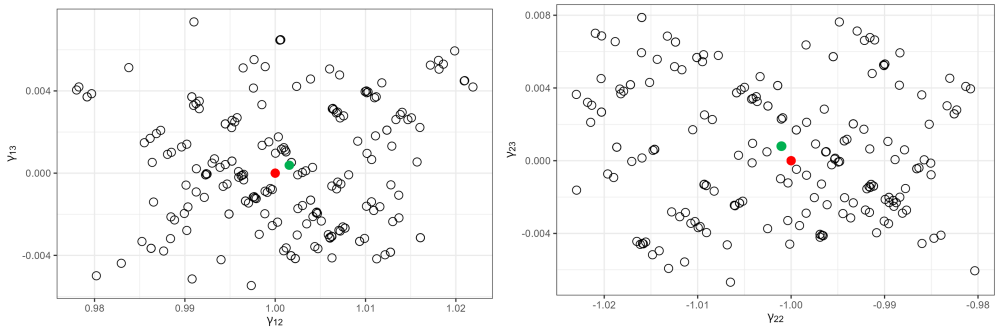
Empirical Model specification results under segmented models with regimes from $K_0 = 4$ to $K_0 = 1$ under 500 times replications. The performances were evaluated by the empirical average of the estimated number of regimes $\widehat{K}$, the discrepancy between the true regimes and the estimated regimes $D(\mathcal{R}, \widehat{\mathcal{R}})$ and the L_2 estimation error of regression coefficients $D(\mathcal{B}, \widehat{\mathcal{B}})$. The penalty parameter λ_T in the model selection criterion was set as $\lambda_T \in \{5, 5\log(T), 5\log^2(T)\}$. The numbers inside the parentheses are the standard errors of the simulated averages.

Model	T	$\lambda_T = 5$			$\lambda_T = 5\log(T)$			$\lambda_T = 5\log^2(T)$		
		$\widehat{K}$	$D(\mathcal{R}, \widehat{\mathcal{R}})$	$D(\mathcal{B}, \widehat{\mathcal{B}})$	$\widehat{K}$	$D(\mathcal{R}, \widehat{\mathcal{R}})$	$D(\mathcal{B}, \widehat{\mathcal{B}})$	$\widehat{K}$	$D(\mathcal{R}, \widehat{\mathcal{R}})$	$D(\mathcal{B}, \widehat{\mathcal{B}})$
Model (2.1) ($K_0 = 4$)	200	4.00 (0.00)	0.03 (0.02)	0.61 (0.12)	3.99 (0.08)	0.03 (0.04)	0.62 (0.16)	2.78 (0.87)	0.87 (0.91)	2.24 (1.05)
	400	4.00 (0.00)	0.01 (0.01)	0.41 (0.08)	4.00 (0.00)	0.01 (0.01)	0.41 (0.08)	3.92 (0.27)	0.05 (0.13)	0.53 (0.43)
	800	4.00 (0.00)	0.01 (0.00)	0.29 (0.05)	4.00 (0.00)	0.01 (0.00)	0.29 (0.05)	4.00 (0.00)	0.01 (0.00)	0.29 (0.05)
	1600	4.00 (0.00)	0.00 (0.00)	0.20 (0.04)	4.00 (0.00)	0.00 (0.00)	0.20 (0.04)	4.00 (0.00)	0.00 (0.00)	0.20 (0.04)
Model (6.1) ($K_0 = 3$)	200	3.44 (0.50)	0.12 (0.11)	0.50 (0.11)	3.00 (0.00)	0.02 (0.02)	0.48 (0.11)	2.85 (0.38)	0.13 (0.30)	0.75 (0.69)
	400	3.39 (0.49)	0.10 (0.11)	0.34 (0.07)	3.00 (0.00)	0.01 (0.01)	0.33 (0.07)	3.00 (0.00)	0.01 (0.01)	0.33 (0.07)
	800	3.33 (0.47)	0.08 (0.11)	0.23 (0.05)	3.00 (0.00)	0.01 (0.00)	0.22 (0.05)	3.00 (0.00)	0.01 (0.00)	0.22 (0.05)
	1600	3.33 (0.47)	0.08 (0.11)	0.16 (0.03)	3.00 (0.00)	0.00 (0.00)	0.16 (0.03)	3.00 (0.00)	0.00 (0.00)	0.16 (0.03)
Model (6.2) ($K_0 = 3$)	200	3.00 (0.00)	0.02 (0.01)	0.47 (0.12)	3.00 (0.00)	0.02 (0.01)	0.47 (0.12)	2.71 (0.46)	0.19 (0.27)	0.97 (0.80)
	400	3.00 (0.00)	0.01 (0.01)	0.31 (0.07)	3.00 (0.00)	0.01 (0.01)	0.31 (0.07)	3.00 (0.00)	0.01 (0.01)	0.31 (0.07)
	800	3.00 (0.00)	0.01 (0.00)	0.22 (0.05)	3.00 (0.00)	0.01 (0.00)	0.22 (0.05)	3.00 (0.00)	0.01 (0.00)	0.22 (0.05)
	1600	3.00 (0.00)	0.00 (0.00)	0.15 (0.03)	3.00 (0.00)	0.00 (0.00)	0.15 (0.03)	3.00 (0.00)	0.00 (0.00)	0.15 (0.03)
Model (6.3) ($K_0 = 2$)	200	3.38 (0.59)	0.14 (0.11)	0.35 (0.10)	2.03 (0.17)	0.01 (0.01)	0.30 (0.08)	2.00 (0.00)	0.01 (0.01)	0.30 (0.08)
	400	3.54 (0.51)	0.13 (0.11)	0.24 (0.07)	2.01 (0.08)	0.01 (0.01)	0.20 (0.05)	2.00 (0.00)	0.01 (0.00)	0.20 (0.05)
	800	3.53 (0.53)	0.12 (0.11)	0.16 (0.04)	2.00 (0.06)	0.00 (0.00)	0.14 (0.04)	2.00 (0.00)	0.00 (0.00)	0.14 (0.04)
	1600	3.50 (0.55)	0.13 (0.12)	0.12 (0.03)	2.00 (0.00)	0.00 (0.00)	0.10 (0.03)	2.00 (0.00)	0.00 (0.00)	0.10 (0.03)
Model (6.4) ($K_0 = 2$)	200	2.93 (0.80)	0.23 (0.18)	0.37 (0.10)	2.00 (0.04)	0.02 (0.02)	0.34 (0.11)	2.00 (0.00)	0.02 (0.01)	0.34 (0.11)
	400	2.80 (0.76)	0.20 (0.18)	0.25 (0.07)	2.00 (0.00)	0.01 (0.01)	0.24 (0.07)	2.00 (0.00)	0.01 (0.01)	0.24 (0.07)
	800	2.68 (0.70)	0.17 (0.17)	0.17 (0.05)	2.00 (0.00)	0.00 (0.00)	0.16 (0.05)	2.00 (0.00)	0.00 (0.00)	0.16 (0.05)
	1600	2.70 (0.69)	0.18 (0.17)	0.15 (0.14)	2.00 (0.06)	0.01 (0.03)	0.14 (0.14)	2.00 (0.00)	0.01 (0.03)	0.14 (0.14)
Model (6.5) ($K_0 = 1$)	200	1.98 (0.70)	0.28 (0.19)	0.17 (0.07)	1.04 (0.01)	0.02 (0.00)	0.13 (0.05)	1.00 (0.00)	0.00 (0.00)	0.13 (0.05)
	400	1.93 (0.68)	0.27 (0.19)	0.12 (0.04)	1.02 (0.00)	0.01 (0.00)	0.09 (0.03)	1.00 (0.00)	0.00 (0.00)	0.09 (0.03)
	800	1.84 (0.59)	0.25 (0.18)	0.08 (0.03)	1.00 (0.00)	0.00 (0.00)	0.07 (0.02)	1.00 (0.00)	0.00 (0.00)	0.07 (0.02)
	1600	1.85 (0.64)	0.25 (0.19)	0.06 (0.02)	1.00 (0.00)	0.00 (0.00)	0.05 (0.02)	1.00 (0.00)	0.00 (0.00)	0.05 (0.02)

H.2. Multiple solutions selected by the MIQP. In our estimation procedure, it is required to produce multiple solutions for $\boldsymbol{\gamma}$ and then take their averages to approximate the centroid of the least squares set $\widehat{\mathcal{G}}$. In this part, we demonstrate the performance of such an approximation by the following simulation.

The data generation process for $\{(Y_i, \boldsymbol{X}_i, \boldsymbol{Z}_i)\}_{i=1}^{T}$ was the same as the independence setting as that in Section 7.1. The sample size used in this simulation was $T = 800$. The true splitting coefficients were $\boldsymbol{\gamma}_{10} = (1, 1, 0)^{\mathrm{T}}$ and $\gamma_{20} = (1, -1, 0)^{\mathrm{T}}$, respectively. By setting the parameters *SolutionNumber* = 200 and *PoolGap* = 0 in the MIQP solver in GUROBI, we obtained 200 solutions whose objective values all attained the minimum, which ensured that these solutions were selected from the $\widehat{\mathcal{G}}$. Figure S3 displays that the selected values were nearly uniformly distributed, and their averages approximated to the true values colored in red and the center of $\widehat{\mathcal{G}}$.

Fig S3: Distributions of the selected 200 optimal solutions for the splitting coefficients. The first elements of γ_1 and γ_2 were omitted since they were normalized as 1. The true values were indicated in red, and the averages of the multiple solutions were indicated in green.



APPENDIX I: ADDITIONAL CASE STUDY RESULTS

The following Table S4 reports basic summary statistics of the involved variables in the training and testing sets.

TABLE S4

Sample means of the training and testing sets of the meteorological variables. The numbers inside the parentheses are the sample standard deviations and the numbers inside the square brackets are the sample correlations with the covariates and $PM_{2.5}$.

Season	$PM_{2.5}$	TEMP	DEWP	PRES	WD	IWS	log(BLH)	RAIN
Training sets								
Spring	48.31	15.87	-1.47	1008.09	3.51	5.89	5.39	0.09
	(49.00)	(8.05)	(9.40)	(6.78)	(1.28)	(8.95)	(1.79)	(0.84)
		[-0.21]	[0.22]	[-0.10]	[0.06]	[-0.20]	[-0.25]	[-0.06]
Summer	38.70	27.34	16.46	998.76	3.40	4.27	5.31	0.46
	(27.08)	(4.41)	(5.50)	(4.34)	(1.33)	(7.53)	(1.62)	(2.70)
		[0.01]	[0.52]	[-0.12]	[0.02]	[-0.19]	[-0.10]	[-0.01]
Fall	49.93	15.42	5.82	1013.51	3.14	3.99	4.87	0.15
	(36.19)	(9.34)	(9.81)	(8.00)	(1.43)	(7.43)	(1.53)	[1.99]
		[0.04]	[0.26]	[-0.25]	[-0.07]	[-0.19]	[-0.12]	[-0.07]
Winter	58.77	0.07	-14.62	1021.05	3.26	4.89	4.58	0.00
	(56.65)	(5.03)	(7.07)	(6.68)	(1.29)	(8.46)	(1.56)	(0.01)
		[-0.03]	[0.57]	[-0.40]	[0.11]	[-0.27]	[-0.33]	[-0.01]
Testing sets								
Spring	54.99	16.87	0.82	1005.28	3.64	11.59	5.46	0.09
	(42.81)	(6.33)	(10.29)	(6.09)	(1.24)	(20.94)	(1.74)	(0.84)
		[-0.28]	[0.54]	[-0.57]	[-0.01]	[-0.33]	[-0.04]	[-0.02]
Summer	41.61	26.96	17.16	997.93	3.31	4.95	5.41	0.32
	(29.42)	(3.95)	(4.16)	(3.33)	(1.28)	(8.08)	(1.50)	(1.21)
		[0.06]	[0.57]	[0.30]	[0.07]	[-0.29]	[-0.02]	[-0.01]
Fall	37.77	13.75	4.25	1014.92	3.25	4.50	4.89	0.17
	(32.64)	(7.74)	(10.71)	(6.06)	(1.40)	(11.34)	(1.47)	(1.79)
		[-0.09]	[0.16]	[-0.12]	[-0.16]	[-0.15]	[-0.11]	[-0.02]
Winter	56.48	-0.31	-14.61	1021.99	3.18	5.09	4.67	0.01
	(83.69)	(4.14)	(6.51)	(4.86)	(1.25)	(7.75)	(1.56)	(0.08)
		[-0.15]	[0.35]	[-0.36]	[0.05]	[-0.22]	[-0.32]	[-0.02]

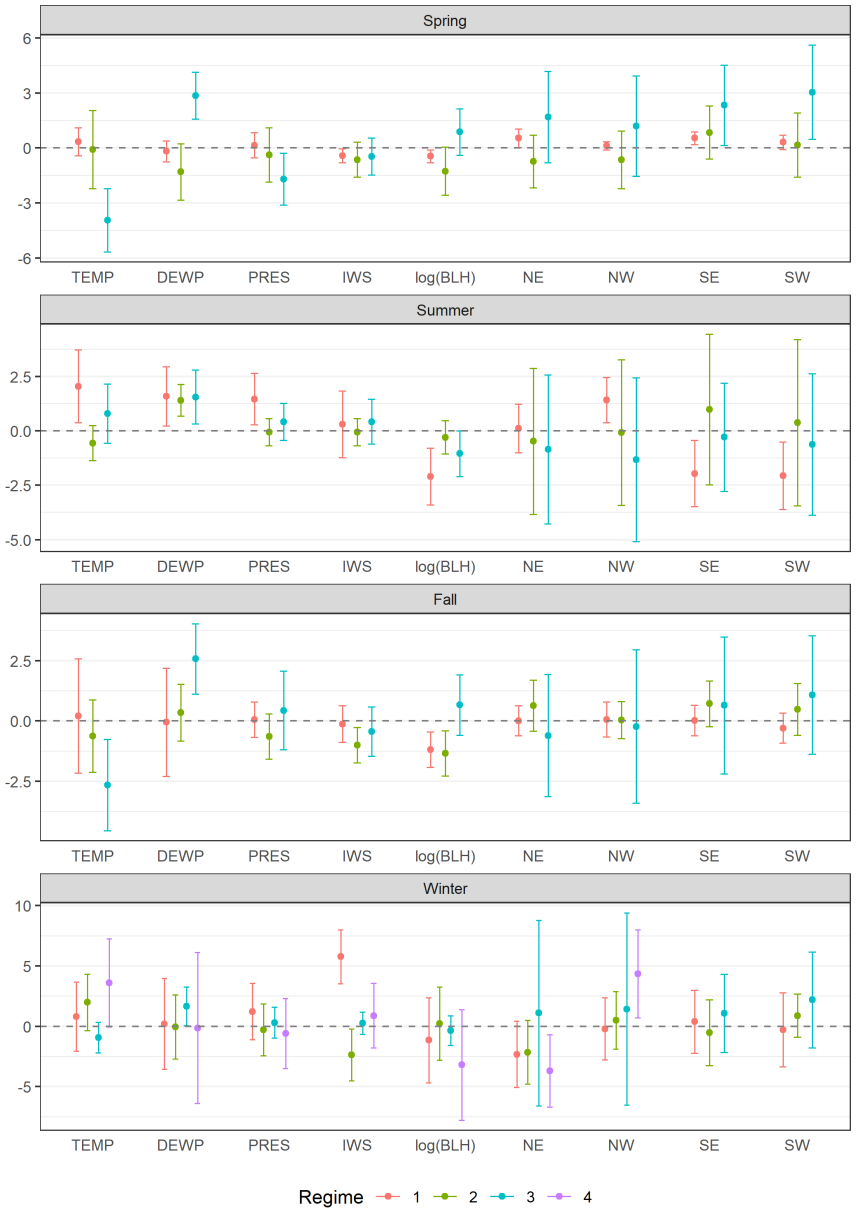
Table S5 reports some important statistics of each estimated regimes for the four seasons, including thes ample sizes, the fitting RMSEs and the sample means of $PM_{2.5}$ and the regression covariates of the estimated regimes.

TABLE S5

The sample sizes, the fitting RMSEs and the sample means of $PM_{2.5}$ and the regression covariates in the four seasons. The numbers inside the parentheses are the sample standard deviations of the sample means above them. RAIN was not included in seasons except for summer since their precipitation was rather sparse.

		T	RMSE	PM2.5	TEMP	DEWP	PRES	log(BLH)	RAIN
Spring	1	119	10.4	67.7	15.5	10.9	1006.7	5.6	
				(40.0)	(5.5)	(4.1)	(5.9)	(1.3)	
	2	793	12.7	61.5	15.9	2.6	1006.6	5.0	
				(55.7)	(8.8)	(7.0)	(6.6)	(1.7)	
	3	528	10.6	23.7	15.9	-10.5	1010.6	5.9	
				(23.0)	(7.4)	(4.7)	(6.5)	(1.8)	
Summer	1	180	9.4	61.9	28.6	23.6	995.9	4.2	0.12
				(37.3)	(3.4)	(1.1)	(3.2)	(5.6)	(0.6)
	2	910	8.4	42.6	27.1	17.9	998.6	2.7	0.2
				(22.8)	(4.4)	(2.9)	(4.2)	(3.2)	(1.2)
	3	343	4.7	16.1	27.1	8.8	1000.6	8.3	8.6
				(10.6)	(4.6)	(3.4)	(4.3)	(13.0)	(10.1)
Fall	1	252	9.2	61.1	15.7	13.3	1011.3	3.8	
				(36.9)	(5.1)	(4.2)	(5.1)	(0.9)	
	2	738	8.9	53.7	18.5	9.4	1011.4	5.0	
				(36.2)	(9.1)	(6.2)	(6.4)	(1.5)	
	3	448	9.1	37.4	10.2	-4.4	1018.3	5.3	
				(32.1)	(9.4)	(8.8)	(9.5)	(1.6)	
Winter	1	288	16.3	94.4	3.0	-9.7	1018.0	5.0	
				(62.1)	(5.2)	(6.8)	(5.8)	(1.7)	
	2	194	11.2	54.9	1.5	-16.2	1021.1	5.2	
				(43.3)	(6.0)	(5.7)	(7.4)	(1.7)	
	3	760	11.7	34.5	-2.0	-21.6	1025.9	4.8	
				(45.0)	(4.9)	(6.6)	(7.0)	(1.6)	
	4	157	15.8	71.6	-3.6	-16.2	1022.2	3.5	
				(55.3)	(4.4)	(5.3)	(7.3)	(1.0)	

Fig S4: Estimated regression coefficients (indicated by dots) and their 95% confidence intervals (indicated by bars) of each regime. The estimated coefficients of the Lag term were all significantly above 0 and thus not reported in this figure.

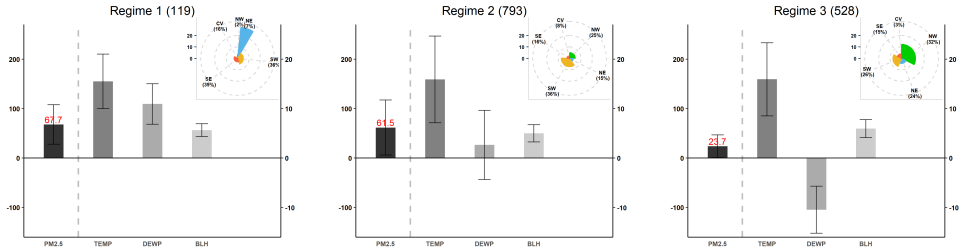


The following Figure S5 displays the estimated meteorological regimes on $PM_{2.5}$. It shows that in spring, for instance, Regime 1 had the highest DEWP and the highest proportion of CV among the three regimes, which is a known condition to encourage secondary generation of $PM_{2.5}$ and to constitute a unfavourable atmospheric diffusion condition, and thus resulted in high $PM_{2.5}$. Regime 2 had reduced percentages of CV and lower DEWP level compared to Regime 1, which alleviated the polluting level and led to better diffusion of $PM_{2.5}$ and can be regarded as a transitional state from either the high pollution to low pollution or vice versa. In Regime 3, the northerly winds occupied the leading position and DEWP was significantly lower. It is noted that the northerly wind brings cleaner and cooler air from the north, and under such circumstances the $PM_{2.5}$ concentration could be effectively reduced via the removal process at a lack of secondary generation. Therefore, Regime 3 represented a cleaning regime.

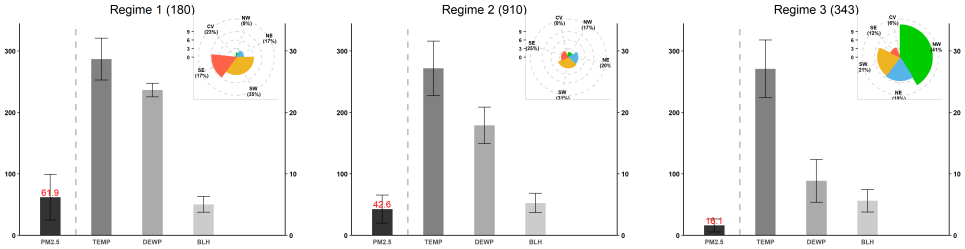
It is found that the regime-splitting for summer and fall shared the same pattern with the spring, namely Regime 1 with high $PM_{2.5}$ level accompanied by a large proportion of CV and high DEWP, indicating an air stagnation; Regime 2 is a transitional regime which had reduced DEWP and increased winds with about 50% southerly winds; and Regime 3 (cleaning) tended to had significantly large amount of strong northerly, in particular northwesterly wind and low DEWP, which are known favorable conditions to lower the $PM_{2.5}$. For winter, Regime 1 was still the most polluting regime and Regime 3 represented the cleaning regimes as the other seasons. However, the transitional regime was divided to two regimes: Regimes 2 and 4 with dominated wind directions being southeasterly and southwesterly wind, respectively, representing two different transitional modes. Regime 4 had more southwesterly wind which would bring the accumulated $PM_{2.5}$ along the foot of Taihang Mountain to Beijing, bringing in more transported air pollutants. As validated in Figure S5, Regime 4 of winter indeed had heavier $PM_{2.5}$ than Regime 2.

Fig S5: For each regime, the bars indicate sample means of $PM_{2.5}$ (scales on the left side), TEMP, DEWP and $\log(\text{BLH})$ (scales on the left side) and the lengths of error bar are twice of the sample deviations. The wind rose plots displays distribution of wind directions (via width of angles) and average speed (via length of radius). Sample sizes of each regime are reported in the parentheses of its subtitle.

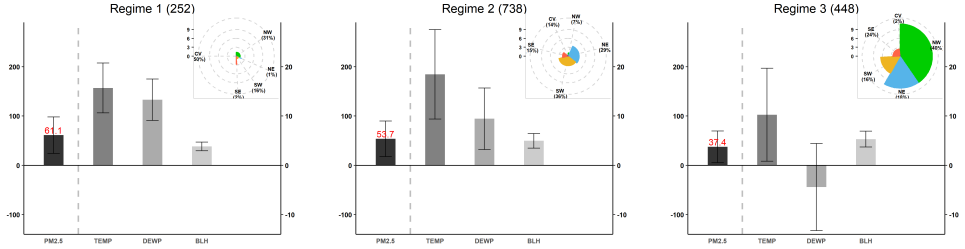
(a) Spring



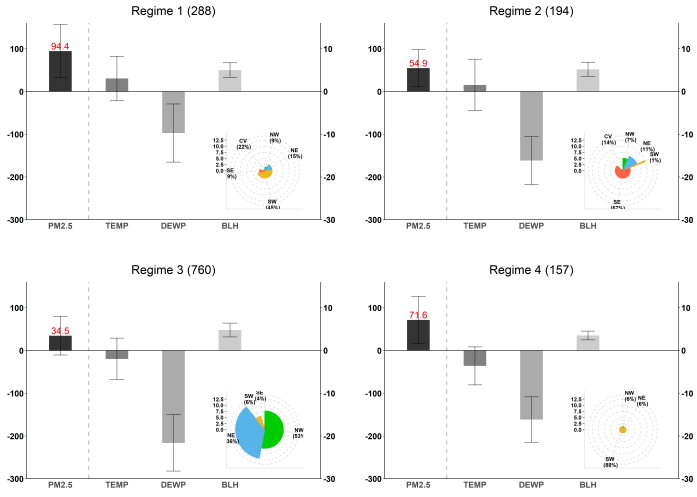
(b) Summer



(c) Fall



(d) Winter



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